THE PRINCIPLES OF THE DIFFERENTIAL AND INTEGRAL CALCULUS;

AND THEIR

APPLICATION TO GEOMETRY.

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INTRODUCTORY NOTE.

The design, in the following pages, is to exhibit the principles of the Differential and Integral Calculus, and to apply those principles to Geometry. The Differential Calculus was in its origin geometrical, and employed by the early writers on the subject as an instrument for the extension of Geometry, and sciences reducible to Geometry. Afterwards in the hands of La Grange and others, it became a field for analysis, and was considered to consist essentially in determining algebraically the differential coefficients of functions. In the following treatise, the principles of this Calculus are examined in reference to Geometry, and elucidated by their application to that science.

Leibnitz, and the early writers on the Calculus, represented by lines the difference of two quantities where that difference was infinitely small. Subsequent analysts found fault with this plan, and insisted on the method of *limits*, as the only correct principle from which the mind could advance in investigations by means of the Calculus. In the present treatise, the *language* of Leibnitz has been retained, but (as it was indeed used by that Geometrician,) only for the sake of convenience. The principle of limits has become too well recognized by able writers on the subject, to be now departed from, and it is to be distinctly understood that in employing in the present volume, the language and the reasoning in respect of infinitely small quantities, or indefinitely small quantities, convenience and clearness of *geometrical* conception have been the only motives. We have been the more induced to adopt this language, from the observation, that the writers who studiously adhere to the language of limits in their discussion of the principles and applications of the Calculus, often make the consideration of indefinitely small quantities the *directrix*

of their reasoning. Little is therefore really gained by adhering rigorously either to the language of limits or to the language of indefinitely small quantities. No error can be introduced by employing the latter when it is understood to be used merely for convenience ; for when once a clear geometrical conception is attained, the transit from indefinitely small quantities to limits is a short journey, and readily made. We will point out in supplementary propositions, the mode of investigation by the principle of limits.

The work here offered to the public, was composed from time to time while the author was engaged in giving instruction in the higher mathematics, and embodies the results of his experience in communicating the elements, and applications of this science. The Calculus is here presented as it was taught in the recitation room, viz. as a branch, or extension of Geometry. The author has not attempted, except perhaps in one or two cases, to attach any other than geometrical significancy to the results of Differentiation and Integration. Nor is it necessary to apologise here for so doing. For the branches of Physical science which have been subjected to the calculations of pure Mathematics, are those which can, by some arbitrary consideration, be reduced within the geometrical relations. For example, in Statics, forces are represented by lines, by simply making the lines proportional to the forces. The moment this arbitrary consideration is introduced, the relations of forces become the relations of lines or surfaces, and Statics is reduced to Geometry. In Dynamics also, the elements of motion, viz. Time, Space, and Force are represented by lines, and immediately the dynamical relations of Time, Space, and Force, are ascertained by ascertaining the geometrical relations of lines, or surfaces. This is the real basis of Physical Mathematics. Hence facility in the application of the Calculus to Geometry disposes of much of the difficulty of reducing Physics to the pure Mathematics, and enables the student to enter with ease, upon those physical investigations which depend upon a proper conception of the geometrical relations.

If therefore, the following pages extend the student's knowledge of Geometry, and enable him to perceive, clearly and correctly, the spirit and mode of investigating geometrical relations by means of

the Differential and Integral Calculus, the author's aim will be attained.

The division into distinct propositions, throughout the work, is a division of convenience, which has some disadvantages it is true, but these are more than compensated by the fact, that such a division directs the attention of the student definitely to a single point. In the applications to Geometry, the propositions are as general and comprehensive as consistency with clearness admitted. The plan has been to solve each proposition in its general form, and elucidate it by one or two particular cases. This plan is preserved throughout the Differential Calculus with the exception of a few propositions on the subject of consecutive surfaces, as Prop. LXI., LXII., and LXIII. If the book had been intended merely for adepts, a more comprehensive plan would have been adopted, but in that case clearness of geometrical elucidation, and consequently, adaptation to the wants of the student could not have been retained.

As no one undertakes the study of the Calculus without a competent knowledge of Algebra and Analytical Geometry, it seemed useless to dwell upon the algebraical details of addition, elimination, &c. To do so would increase the size of the work, without increasing its valuable matter, inasmuch as any student can readily perform such operations. The difficulty in studying most works on the Calculus, as well as on Analytical Geometry, arises in part from these very algebraic details. When presented to the eye of the student in the body of the work they cause him to lose sight of principles, and engross his attention to the exclusion of more valuable matters. For the same reason we have not entered much into the details and processes which appertain properly to books on Analytical Geometry.

The theory of consecutive lines, commencing with Prop. XXXVII., seemed to possess peculiar advantages in a geometrical point of view, and is rendered subservient to many elucidations, both in the Differential and Integral Calculus. The theory of consecutive surfaces likewise presented geometrical advantages, and is introduced at Prop. LIX.

The Geometry of Curve Surfaces commences at Prop. XLV., and

will be found as extensive as is usually studied in the Colleges or Universities in the United States. With proper preparation, the Geometry of Curve Surfaces presents perhaps fewer difficulties than are encountered in the Geometry of Curves; especially is this the case, when the Geometry of Curves is first mastered.

In the Integral Calculus the rules for integrating are as concise as clearness, and the nature of the subject seemed to admit. It has often been remarked that no branch of Mathematics requires more skill in algebraic processes, than the Integral Calculus. The student's progress in this, is dependent altogether upon the skill which he possesses in algebraic operations.

The present treatise contains the principles for integrating partial differential equations of the first and second orders, and higher degrees of the first order. The Geometry of the Integral Calculus, will be found more detailed than in most books on the subject.

Every writer, who discusses topics already investigated, must borrow more or less of the light of others. We acknowledge the advantages received from the study of Euler, La Croix, &c., but it is only that advantage, which accrues to the writer on any science, who is acquainted with what his predecessors, or cotemporaries, have done on the same subject

With these remarks, the present treatise is committed to the public, with the hope that it may meet the approbation of Professors, and facilitate the labors of students, in acquiring a knowledge of the principles and Geometry of the Calculus.

Easton, Pa., September 1st, 1844.

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DIFFERENTIAL CALCULUS.

DEFINITIONS.

RATIO is the quotient that arises from dividing one quantity by another. Thus the ratio of a to b is $a \div b$.

We will, in general, use the word *line* to designate either a straight line or a curve line.

The first letters of the alphabet, a, b, c, &c., are in general employed to designate constant quantities, and the last letters, w, x, &c., to designate variable quantities.

The object of the Differential Calculus is to determine, by means of Ultimate Ratios, the properties and relations of lines, and the properties and relations of surfaces, and the relations of lines to surfaces. It is a branch, or rather a continuation of Analytical Geometry, and presupposes an acquaintance with the elements of that science.

The nature of the Ultimate Ratios employed in the Calculus will be better elucidated by the first of the following Propositions, than by any à priori discussion.

Equations and Quantities are usually divided into Logarithmic, Circular, and Algebraic.

If we take, as an example, the equation of a plane curve, which involves the two variable co-ordinates and constants, we may define a logarithmic equation or quantity to be one that involves either or both of the variables in a logarithm. Thus the equation

$$y = a \log. x + b$$

is a logarithmic equation, the variable x being involved in a logarithm. In like manner, the following are logarithmic equations:

$$y = \log. (a + x), \qquad \log. y = \log. (a + x) + b.$$

But the equations

$$y = x \log. a, \qquad y = (x + b) \log. a$$

are not logarithmic equations ; for though they contain logarithms, yet the variables x and y are not involved in the logarithms.

A circular equation or quantity, is one involving either or both of the variables in a trigonometrical line, or arc of a circle. Thus the equations

$$y = \sin. x + b, \qquad y = \tan. (x + h)$$

are circular equations, the variable x being, in each, involved in a trigonometrical line.

Logarithmic and circular quantities are frequently called *transcendental*, as distinguished from algebraic quantities.

Algebraic equations, and quantities, are those that do not contain either of the variables involved in a logarithm, trigonometrical line, or arc of a circle. Thus

$$y = ax^3 + b, \qquad y = ax + x^2, \qquad y = mx$$

are algebraic equations, and

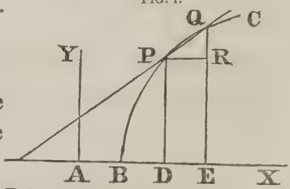
$$y = x \tan. c, \qquad y = x^2 \sin. c,$$

are also algebraic equations ; for though they contain trigonometrical lines, viz., tan. c and sin. c, yet c being constant, there is no involution of variable quantities in the trigonometrical lines.

PROPOSITION I.

Given the algebraic or logarithmic equation of a curve, determine the Ultimate Ratio of the increments of its co-ordinates.

FIG. 1.



Let the equation of the curve B P, referred to the axes A X and A Y be

(1) $\quad y = ax^3 + b.$

For any point P on this curve, the co-ordinates x and y of equation (1) are A D and D P.

Suppose that in (1) x the abscissa of P be increased by D E, then putting D E $= h$, and Q E $= y'$, equation (1) becomes

(2) $y' = a(x + h)^3 + b$,

which is the equation of B P for the point Q.

Expanding the second side of (2) by the Binomial Theorem, we have

(3) $y' = ax^3 + 3ax^2h + 3axh^2 + ah^3 + b$.

Subtract equation (1) from (3), and we have

(4) $y' - y = 3ax^2h + 3axh^2 + ah^3$.

Divide (4) by h, and we have

(5) $$\frac{y'-y}{h} = 3ax^2 + 3axh + ah^2$$

In (4) and (5) it is evident that $y'-y =$ Q R, the increment of the ordinate, in going from P to Q on the curve B P, and h is the increment D E of the abscissa between the same points. The first side of (5), therefore, expresses the ratio of the increment of the ordinate to the increment of the abscissa, consequently the second side of (5) expresses the same ratio.

If now we suppose the increment D E, or h, of the abscissa to decrease, the corresponding increment Q R of the ordinate will also decrease, and regarding the second side of (5) as expressing the ratio of these increments, it is obvious that their ratio also decreases as h decreases.

It is further obvious, that the ratio expressed by the second side of (5) cannot become less than $3ax^2$, for that is what the second side of (5) becomes when h is zero.

This value, $3ax^2$, is called the Ultimate Ratio, or LIMIT of the ratio, because it is the *limit* beyond which the ratio of the increments cannot diminish. Equation (5), regard being had only to the second side of it, becomes for the Ultimate Ratio, or when h is zero,

(6) $$\frac{y'-y}{h} = 3ax^2$$

The same result may be written by observing that if the increment h in (5) be taken indefinitely small, the terms containing h become indefinitely small, and may, when compared to the terms that do not contain h, be considered as zero, and consequently may be rejected from the equation.

In (5), when the increment h is indefinitely small, the difference $y'-y$ of the ordinates is written dy, which is called the *differential*

of y, and the increment h of the abscissa is written dx, which is called the *differential of* x.

If therefore, we put

(7) $y'-y = dy$, and $h = dx$,

equation (6) may be written,

(8) $\dfrac{dy}{dx} = 3ax^2$

which is the Ultimate Ratio, or *limit* of the ratio of the increments of the co-ordinates of the curve whose equation is (1).

Rigid analysis derives (8) from (5), on the supposition that in (5) the increment h of the abscissa decreases, and finally becomes zero, in which case (5) becomes (8) and $\dfrac{dy}{dx}$ is used as a mere symbol to denote the Ultimate Ratio, $\dfrac{dy}{dx}$ being in reality $\dfrac{o}{o}$. But inasmuch as the rules for differentiating, and the geometrical applications of Ultimate Ratios are more readily understood by regarding the increments of the ordinate and abscissa as indefinitely small, we will call these increments in their ultimate state, ***indefinitely small quantities.*** To prevent misunderstanding, it may be proper to repeat an observation made in the Introductory Note, that this language is adopted for the sake of convenience, and not for the purpose of indicating a departure from the well established ideas in relation to Ultimate Ratios.

The student who is familiar with the process, in Analytical Geometry, of determining the equation of a tangent line to a curve, by first obtaining the equation of a secant line, and then supposing the secant points to coincide, cannot fail to observe the similarity, and even identity of that process with the one just detailed. Indeed the Ultimate Ratio, as will be shown in the next Proposition, expresses the tangent of the angle which the tangent line to a plane curve makes with the axis of abscissas. The tangent of this angle is obtained, in Analytical Geometry, by the consideration of the secant line.

But let us return to equation (8). If (8) be cleared of fractions, we have

(9) $dy = 3ax^2\,dx$,

which is called, *the differential* of equation (1).

If, instead of (1), the equation of the curve B P were

(10) $y = ax^n + b$,

then by giving to x an increment $h =$ D E, and putting y' for the corresponding ordinate Q E, equation (10) becomes

(11) $y' = a\,(x + h)^n + b$.

Expand (11) by the Binomial Theorem, and we have

(12) $y' = ax^n + anx^{n-1}h + a.\,n.\,\frac{n-1}{2}.\,x^{n-2}h^2 + \&c., + b.$

Subtract (10) from (12), divide by h, and we have

(13) $\dfrac{y'-y}{h} = anx^{n-1} + a.\,n.\,\frac{n-1}{2}.\,x^{n-2}\,h + \&c.$

As every term after the first, on the second side of (13), contains h, it is obvious that when h is made zero, or indefinitely small, the second side of (13) reduces to anx^{n-1}, and then by the notation at (7), (13) becomes

(14) $\dfrac{dy}{dx} = anx^{n-1}$,

or clearing of fractions,

(15) $dy = anx^{n-1}dx$,

which is the differential of (10), and (14) is the differential coefficient of (10).

In these cases, the development of (2) and of (11) is effected by the Binomial Theorem. In that Theorem, as is shown in most books on Algebra, the rules for the development are the same, whether the exponent of the binomial be integral or fractional, positive or negative. Consequently we may regard the process which derived (9) from (1), or (15) from (10), to be as general as the Binomial Theorem, and we know that the development may be effected by that Theorem, whenever the equation of the curve is algebraic. Hence the process of obtaining the Ultimate Ratio in an algebraic curve is fully exhibited from (10) to (14). When the equation is transcendental, the development will be effected by other processes, as will be shown hereafter. We might, as is done in most books on the subject, write down a general form for the development of the expression, after the abscissa receives the increment, but perhaps a clearer idea of the processes can be obtained by first introducing the actual developments, as the student is acquainted with them in his books on Algebra, Trigonometry, &c.

In equation (10), if n be negative, the second term of (12), by the principles of the Binomial Theorem, is also negative, and (15) is negative. Hence if we have the equation

$$y = ax^{-n},$$

the differential corresponding to (15) is

$$dy = -anx^{-n-1}dx.$$

By comparing (9) to (3), or (15) to (12), we observe that (since $dx = h$) the second side of (9) is the second term of the development of the binomial (2), and the second side of (15) is the second term of the binomial development (12). Hence we see that the Ultimate Ratio, (14), viz. anx^{n-1}, is the coefficient of the first power of the increment h of the abscissa in the development (12). Hence the Ultimate Ratio is called the *Differential Coefficient*, a term which we will hereafter employ as synonymous with Ultimate Ratio.

By comparing (10) and (15), we observe that the second side of (15) may be produced from (10,) by multiplying (10) by the exponent of the variable x, decreasing the exponent by unity, and multiplying by dx. This may be expressed by the following

RULE I.

To differentiate a quantity raised to a power, multiply by the exponent, diminish the exponent by unity, and multiply by the differential of the quantity.

Thus the differential of cx^4 is $4\,cx^3dx$. Here x is the quantity raised to the power 4, hence we multiply the quantity raised to the power by 4, diminish the power by 1, and multiply by dx. The quantity b in (10) disappears when we subtract (10) from (12), and is not found in (15). Hence we have

RULE II.

A constant added disappears in differentiating, or which is the same thing, *the differential of a constant is zero.*

The factor a in (10) remains in the differential (15). Hence,

RULE III.

In differentiating the product of a constant factor and variable the constant factor remains unchanged.

As an application of these Rules, take the following

Examples.

$$y = nx^5 + c.$$

The differential of this is

$$dy = 5\,nx^4dx,$$

the constant factor n remains by Rule III, and the constant quantity c, which is added, disappears by Rule II.

Differentiate the following equations :

$$y = ax^{\frac{3}{2}}, \text{ gives } dy = \tfrac{3}{2}ax^{\frac{1}{2}}dx$$

$$y = ax^{\frac{1}{4}} + c, \text{ gives } dy = \tfrac{1}{4}\,ax^{-\frac{3}{4}}dx$$

$$y = ax^{-\frac{1}{2}} + c, \text{ gives } dy = -\,\tfrac{1}{2}\,ax^{-\frac{3}{2}}dx.$$

If the equation were

$$y = ax + b,$$

then regarding the exponent of x as unity, we have, by the foregoing Rules,

$$dy = 1\,ax^0dx = adx,$$

from which we see, that if the variable rises only to the first power, the differentiation is effected by merely putting the characteristic d before the variable.

If we have an equation of the form

(16) $\quad y = x + z + c,$

where x and z are both variables, then if when y becomes $y + dy$, x becomes $x + dx$, and z becomes $z + dz$, (16) gives

(17) $\quad y + dy = x + dx + z + dz + c,$

subtract (16) from (17), and there remains for the differential of (16),

(18) $\quad dy = dx + dz.$

Comparing (18) and (16), we have

RULE IV.

The differential of the sum of any number of variables is the sum of their differentials.

Rule I. may be extended to a large class of binomial and polynomial expressions, as follows. Suppose we have

(19) $y = (a + bx^4)^n$

Assume

(20) $a + bx^4 = z$

and (19) becomes

$y = z^n$, this differentiated by Rule I. is

(21) $dy = nz^{n-1}dz.$

Differentiate (20), and we have

(22) $dz = 4\,bx^3dx,$

substitute the values of z and dz from (20) and (22) into (21), and we have

(23) $dy = n\,(a + bx^4)^{n-1}4bx^3dx,$

which is the differential of (19).

By comparing (19) and (23) with Rule I., we observe, that if we regard $a + bx^4$ as a single quantity raised to the power n, Rule I. furnishes the direction for differentiating (19), and similar forms.

Ex. 1.—Differentiate the equation

$$y = (ax + bx^3)^n.$$

Here we might put the part in the vinculum equal to a single quantity z, and proceed as was done with (19) to (23), or taking the part in the vinculum as a single quantity, and applying Rule I., we have for the differential,

$$dy = n\,(ax + bx^3)^{n-1}\,(adx + 3\,bx^2dx).$$

Ex. 2.—Differentiate the equations

$$y = (ax - bx^2)^{-n}, \qquad y = (ax + bx^2)^{\frac{1}{n}}.$$

There is another form which may be differentiated by the immediate application of Rule I. Suppose we have

(24) $y = (z + x)^n,$

in which x and z are both variable: assume

(25) $z + x = u$

and (24) becomes

(26) $y = u^n.$

Differentiate this by Rule I., and we have

(27) $dy = nu^{n-1}du.$

Differentiate also (25) by Rule IV., and we have

(28) $du = dz + dx.$

The values of u and du substituted from (25) and (28) into (27), we have

(29) $dy = n(x + z)^{n-1}(dz + dx).$

This is the differential of (24).

Comparing (24) and (29) with Rule I., we observe that if we regard $x + z$ as a single quantity raised to the power n, Rule I. furnishes the direction for differentiating (24), and similar forms.

As an example, take the equation

$$y = (ax + bz)^n.$$

Here we might put the part in the vinculum equal to a single quantity u, and proceed as was done from (24) to (29,) or taking the part in the vinculum as a single quantity, we may, by Rule I. write down the differential, viz.

$$dy = n(ax + bz)^{n-1}(adx + bdz).$$

Differentiate the following equations.

$$y = (a + bx)^{\frac{1}{4}}, \qquad y = (az^2 + bx)^{\frac{1}{n}}, \qquad y = (ax + b)^{-\frac{1}{2}}.$$

When the exponent is $\frac{1}{2}$, we may deduce from Rule I. a convenient rule for differentiating. For, take the equation

(30) $y = x^{\frac{1}{2}}.$

Differentiate this by Rule I. and we have

(31) $dy = \dfrac{dx}{2x^{\frac{1}{2}}}.$

Comparing (30) and (31) we have

RULE V.

The differential of the square root is the differential of the part under the root divided by double the root.

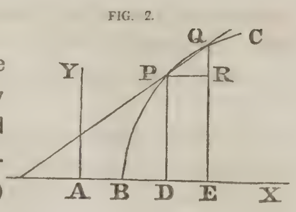
Ex.—Differentiate

$$y = \sqrt{ax}, \qquad \text{gives } dy = \frac{adx}{2\sqrt{ax}}$$

$$y = (2ax - x^2)^{\frac{1}{2}}, \qquad y = (x^2 - z)^{\frac{1}{2}}.$$

Let us now proceed to the second part of the Proposition, viz. to find the differential of a Logarithmic Equation. For this purpose, let

FIG. 2.



(32) $y = \log. x$

be the equation of a curve B Q, where x and y are the co-ordinates of any point P on the curve. If x be increased by D E or h, then representing the corresponding ordinate Q E by y', (32) becomes

(33) $y' = \log. (x + h)$.

If now we could develope the second side of (33) into a series of monomials, we might then, by subtracting (32) from (33) developed, and dividing the remainder by h, obtain the ratio of the increment of the ordinate to the increment of the abscissa, as was done in (10)—(13). But as we cannot readily develope $\log. (x + h)$, the second side of (33), let us first subtract (32) from (33), and we have

(34) $y' - y = \log. (x + h) - \log. x$.

Since the difference of the logarithms of two numbers is the logarithm of the quotient of the numbers, (34) becomes

(35) $y' - y = \log. \left(1 + \frac{h}{x}\right)$

and if this be developed, the result should obviously be the same as if we had first developed (33) and subtracted (32) from the result. Now as is shown in most books on Algebra, if M be the modulus of a system of logarithms, the log. of $1 + n$ is

(36) $\log. (1 + n) = M \left(n - \frac{n^2}{2} + \frac{n^3}{3} - \frac{n^4}{4} + \&c.\right)$.

By making $n = \frac{h}{x}$ (36) becomes

(37) $\log. \left(1 + \frac{h}{x}\right) = M \left(\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \frac{h^4}{4x^4} + \&c.\right)$

This is the development of (35). Substitute it into (35), and dividing by h, we have

(38) $\frac{y' - y}{h} = M \left(\frac{1}{x} - \frac{h}{2x^2} + \frac{h^2}{3x^3} - \frac{h^3}{4x^4} + \&c.\right)$

This is the ratio of the increments of the co-ordinates. To obtain

the limit of this ratio, or the Ultimate Ratio, put h zero, or indefinitely small, and (38) becomes

$$(39) \qquad \frac{y'-y}{h} = \frac{\text{M}}{x}, \text{ or by notation 7, } \frac{dy}{dx} = \frac{\text{M}}{x},$$

This is the Ultimate Ratio sought. In the Naperian, or hyperbolic system of logarithms, $\text{M} = 1$ and (39) becomes

$$(40) \qquad dy = \frac{dx}{x},$$

which is the differential of (32).

Comparing (32) and (40), we have

RULE VI.

To differentiate the logarithm of a quantity, differentiate the quantity and divide by the quantity.

This Rule gives the differential only when the logarithms are taken in the Naperian or hyperbolic system. For any other system of logarithms, we must multiply by the modulus, as is shown in (39).

Ex.—Differentiate

$$y = \log. (a + x).$$

Here $a + x$ is the quantity of which the logarithm is taken; hence by Rule VI. we have

$$dy = \frac{dx}{a + x}.$$

In like manner, differentiate the equations

$$y = a \log. x, \quad \text{gives } dy = \frac{adx}{x},$$

$$y = a \log. (x^2 + bx), \quad \text{gives } dy = \frac{a\,(2xdx + bdx)}{x^2 + bx},$$

$$y = \log. (ax - x^2), \qquad y = n \log. (x^2 - x)^{\frac{1}{2}}.$$

The n^{th} power of a logarithm is usually written by placing the n after the word log. as an exponent. Thus $\log.^n x$, means the n^{th} power of the logarithm of x, and $\log. x^n$ means the logarithm of x^n The form $\log.^n x$ is the same as $(\log. x)^n$. To differentiate a logarithm raised to a power, we combine Rule I. with Rule VI.

Ex.—Differentiate

$$y = \log.^n x, \quad \text{gives } dy = n \log.^{n-1} x . \frac{dx}{x},$$

$$y = \log.^n (a + x), \qquad dy = n \log.^{n-1} (a + x) . \frac{dx}{a+x},$$

$$y = \log.^3 (ax + bx^2), \qquad y = \log.^n (a - bx^m).$$

By the application of Rule VI., and the well known principles of logarithms, we can readily deduce Rules for differentiating a product a fraction, or an exponential. We will examine each of these in order.

First. To differentiate the product of two variables, let us take the equation

(41) $$y = zx,$$

where z and x are both variable.

Take the logarithm of equation (41) and we have

(42) $$\log. y = \log. z + \log. x.$$

Differentiate this by Rule VI. and we have

(43) $$\frac{dy}{y} = \frac{dz}{z} + \frac{dx}{x}.$$

Multiply this by (41) and we have

(44) $$dy = xdz + zdx,$$

which is the differential of (41).

Comparing (41) and (44) we have

RULE VII.

To differentiate the product of two variables, multiply the second by the differential of the first, and the first by the differential of the second, and add the products.

Ex.—Differentiate

$$y = axz + b, \quad \text{gives } dy = axdz + azdx,$$

$$y = ax^2z^3 + b, \quad \text{gives } dy = 2axz^3dx + 3ax^2z^2dz,$$

$$y = z \log. x, \quad \text{gives } dy = \log. x . dz + z . \frac{dx}{x}.$$

The process for differentiating a product of two factors is often conveniently applicable when each factor contains the same variable.

Thus if we have the equation

$$y = x. \log. (a + x)$$

we differentiate it as two factors, x being one factor, and log. $(a+x)$ the other. Hence by Rule VII. the differential is

$$dy = \log. (a + x). dx + \frac{x. dx}{a + x}.$$

Ex.—Differentiate

$y = x^2 \log. (ax + b)$, $y = (a + bx)(m + nx)$, $y = x^n \log.^m x$,

$y = (x + a)(x^2—1)^{\frac{1}{2}}$, $y = (x—x^2)^{\frac{1}{2}} \log. (a + x)$

Proceed in the same manner for the deduction of the Rule for differentiating when the product is that of three or more variables. For if we have

(45) $$y = uzx,$$

by taking the logarithms of (45), differentiating and multiplying the result by (45) we have

(46) $$dy = uzdx + uxdz + xzdu,$$

which furnishes the Rule.

Second. To differentiate a fraction, let us take the equation

(47) $$y = \frac{x}{z}.$$

Take the logarithm of this, and we have

(48) $$\log. y = \log. x—\log. z.$$

Differentiate (48) by Rule VI., and multiplying the result by (47) we have

(49) $$dy = \frac{zdx—xdz}{z^2}.$$

which is the differential of (47).

Comparing (49) and (47) we have

RULE VIII.

To differentiate a fraction, multiply the denominator into the differential of the numerator, from this subtract the numerator into the differential of the denominator, and divide by the square of the denominator.

Ex.—Differentiate

$$y = \frac{ax}{z^2}, \quad \text{gives } dy = \frac{az^2dx - 2axzdz}{z^4},$$

$$y = \frac{x^n}{z^a}, \quad \text{gives } dy = \frac{nx^{n-1}z^a dx - ax^n z^{a-1}dz}{z^{2a}}.$$

If the numerator be constant, its differential is, by Rule II., zero.

Ex. Differentiate

$$y = \frac{a}{x}, \quad \text{gives } dy = -\frac{adx}{x^2},$$

$$y = \frac{a}{x2}, \quad y = \frac{a}{x+b}, \quad y = \frac{a}{x^2 + \log. x}.$$

Third. An exponential is a quantity with a variable exponent. Thus a^x is an exponential. To differentiate an exponential, let us take the equation

(50) $$y = a^x.$$

Take the logarithm of this, and we have

(51) $$\log. y = x \log. a.$$

Differentiate this by Rule VI. and we have

(52) $$\frac{dy}{y} = \log. a. dx.$$

Multiply this by (50), and we have

(53) $$dy = \log. a. a^x. dx,$$

which is the differential of (50).

Comparing (50) and (53), we have

RULE IX.

To differentiate an exponential, multiply together the logarithm of the base, the exponential itself, and the differential of the exponent.

Ex.—Differentiate

$$y = a^{2x}.$$

Here the exponent is $2x$, hence by the Rule

$$dy = \log. a. a^{2x}. 2dx.$$

Differentiate

$$y = (a + c)^{nx}, \quad \text{gives } dy = \log. (a + c). (a+c)^{nx} ndx,$$

$$y = a^{3x}, \quad y = a^{xn}.$$

Rule IX. is deduced on the supposition that the base of the exponential is constant. If the base and exponent be both variable, we

can, by the process by which we passed from (50) to (53), obtain the differential. Thus if we have

(54) $$y = u^x,$$

where u is variable, take the logarithm of the equation, and we have

$$\log. y = x \log. u.$$

The differential of this by Rules VI. and VII. is

$$\frac{dy}{y} = \log. u. dx + x \frac{du}{u}.$$

Multiply this by (54), and we have

$$dy = u^x \left(\log. u. \ dx + x \frac{du}{u}\right),$$

which shows the process.

Differentiate

$$y = (a + x)^x, \qquad y = (a + x^2)^x, \qquad y = (a + u^n)^x.$$

The foregoing Rules are sufficient for the differentiation of algebraic and logarithmic quantities. We might have commenced with the deduction of Rule VI. for differentiating a logarithm, and then have deduced Rule I. from Rule VI. For if we have the equation

(54a) $$y = x^n,$$

by taking the logarithm of this we have

$$\log. y = n \log. x.$$

The differential of this is, by Rule VI.

$$\frac{dy}{y} = n \frac{dx}{x},$$

and multiplying this by the equation (54a), we get

$$dy = nx^{n-1}dx,$$

which is the differential of (54a), and gives Rule I. We preferred, however, introducing the Binomial Theorem to develope equations (1) and (10) after the abscissa had received an increment h, inasmuch as the student is perhaps more familiar with that Theorem than with the Logarithmic Theorem introduced at equation (36), and because the development at (36) is itself deduced by the aid of the Binomial Theorem.

It will be observed that Rules I. and VI. are obtained by the consideration that (10) and (32) are the equations of plane curves

The other Rules are then deduced from these two. *The Curves, however, are merely introduced for the purpose of illustration*, the Rules of differentiating just deduced being in fact algebraically true, and altogether independent of the Curves.

There remains another class of quantities involving trigonometrical lines, and circular arcs. Before, however, deducing the Rules for differentiating these we will give a number of geometrical applications illustrative of the use and signification of differentials and differential coefficients. This will require frequent employment of the foregoing Rules for differentiation. The student will carefully observe the difference between the differential of a quantity and the differential coefficient. Equation (9) is the differential of (1) and equation (8) is the differential coefficient of (1); in like manner (15) is the differential of (10) and (14) the differential coefficient of (10.)

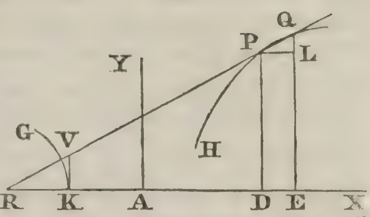
In deducing Rules I. and VI. we gave an arbitrary increment h to one of the variables x in the equation of the curve, and the corresponding increment of the other variable y was deduced from this, and dependent upon it. This may be seen in equations (10) to (15), or in (1) to (9), or in (32) to (34). The variable to which the arbitrary increment was thus given, is called *the independent variable*, and the other is called *the dependent variable.* Thus in the foregoing deduction, x is the independent, and y the dependent variable, we will in general in the equation of a curve take the abscissa x for the independent and the ordinate y for the dependent variable.

In the subsequent propositions the axes of reference are supposed to be rectangular except where it is otherwise specified.

PROPOSITION II.

The differential coefficient deduced from the equation of a plane curve represents the tangent of the angle which the tangent line to the curve makes with the axis of abscissas.

FIG. 3.



Let the equation of the curve H P be

(55) $$y=ax^3+b.$$

At any point P whose co-ordinates are x and y suppose a tangent line R P be drawn. This tangent line may be considered as coinciding with the curve along the indefinitely small distance P Q. The point Q being taken indefinitely near to P, the line P L, which is parallel to the axis A X, represents the increment of the abcissa x, and Q L the increment of the ordinate y. P L and Q L may therefore be taken as the representatives of dx and dy. With the centre R and radius R K=1 describe the arc K G and draw K V tangent to the ark at K.

By similar triangles R K V and P L Q we have

$$\text{R K} : \text{K V} :: \text{P L} : \text{L Q}$$

or since K V is the tangent of R,

(56) $$1 : \tan.\text{R} :: dx : dy$$

from which we have

(57) $$\frac{dy}{dx}=\tan.\text{R}.$$

But the angle at R is the angle which the tangent line, tangent at the point P, or (x, y), makes with the axis of abscissas. Hence (57), which shows that the tangent of this angle equals the differential coefficient, proves the proposition.

Equation (57) is independent of the equation (55), not being deduced from it, consequently the proposition is true whatever be the nature of the plane curve H P. To obtain the tangent of the angle R for any given curve (55) we may therefore deduce the differential coefficient from (55) and its value is the tangent of the angle required.

As a particular example the differential coefficient from (55) is by Rule I.

(58) $$\frac{dy}{dx}=3\,ax^2$$

Consequently for the angle R we have in this particular case

(59) $$\tan.R = 3\,ax^2.$$

In 59 the tangent of R depends upon the value of the abscissa x.

The Rules of Proposition I. are algebraically true, and explain the mode of differentiating algebraic and logarithmic equations; and when differentiated, the differential coefficient may be deduced by the ordinary rules of Algebra. *Proposition II. shows the point of connexion between these algebraic Rules and Geometry*, viz: "The differential coefficient represents the tangent of the angle," &c. From this point of connexion a great system of Geometry may be constructed, inasmuch as almost EVERY GEOMETRICAL INVESTIGATION, BOTH OF LINES AND OF SURFACES, INVOLVES, EITHER DIRECTLY OR INDIRECTLY, THE TANGENT OF THIS ANGLE. The following Propositions, from page 20 to page 59, exhibit the mode of resolving problems in plane Geometry, by aid of the principle established in this Proposition II. At Proposition XLV. commences the application of the same principle to the solution of problems in the Geometry of surfaces. The pure theory, or Rules of differentiation, are exhibited in Propositions I., XXIX., and XXXII., which (if it is preferred) may be first studied; and then the application to Geometry of the principle evolved in Proposition II. may be perused in the other Propositions, as far as time and circumstances admit.

To obtain the differential coefficient from an implicit equation such as $a^2\,y^2 + b^2\,a^2 - a^2\,b^2 = o$, we may either first solve the equation for y and then differentiate, or (which is more convenient in practice) we may first differentiate each term of the equation, and then deduce the differential coefficient by the ordinary rules of algebra.

Before proceeding farther, it may be advantageous to explain a notation that has been found very convenient. If we have any number of curves whose equations are

(A) $$y = ax^2, \quad y = a^x, \quad y = ax - x^2, \quad y = (R^2 - x^2)^{\frac{1}{2}},$$

where y equals an expression into which x enters we may represent them all by the single form.

(60) $$y = \varphi x,$$

which is read, *y equals a function of x*, and signifies that y equals

an expression composed of x and constants. Equation (60) may therefore, be used to designate any curve whose equation is solved for y, and is a general form embracing all such equations as those marked (A.)

Instead of φ in (60) any other character may be employed for the same purpose. Thus,

$$y = \mathrm{F}x, \quad y = \Psi x, \quad y = fx, \quad y = \pi x,$$

would each mean the same as (60) viz. that y equals an expression composed of x and constants.

If the equations of the curves instead of being solved for y as in (A) were of the form,

$$\text{(B)} \quad \begin{cases} x^2 + y^2 - \mathrm{R}^2 = o, & y^2 - px = o, \\ xy - a^2 = o, & ax^2 - xy + y^2 - c^2 = o, \end{cases}$$

i. e. not solved for y, then they may be represented by the form,

$$\text{(61)} \qquad \varphi\,(x,y) = o,$$

which is read, *a function of x y equals zero*, and means that the equation of the curve contains the co-ordinates x y and constants, as in any one of the group B.

Equations of the form A or (60), i. e. solved for y are said to contain y as an *explicit* function of x.

Equations of the form B or (61), i. e. not solved for y are said to contain y as an *implicit* function of x. One quantity is said to be a *function* of another when it depends upon it. Thus in the curve H P, the ordinate P D or y is said to be a *function* of the abscissa A D or x, because if the abscissa be varied the ordinate will also vary, or in equation (60) which is the explicit form of the equation of any plane curve whatever, if x be increased or diminished y will also vary, being connected with x in the equation. Either of the forms (60) or (61) may be used to designate any plane curve whatever, (60) being used when the equation is explicit, and (61) when it is implicit.

In the case of the parabola, (60) is $y = \sqrt{px}$, the explicit form, and (61) is $y^2 - px = o$, the implicit form.

In many of the following propositions we will use the explicit form (60), of the equation of a plane curve, employing it as a mere

symbol to denote the equation of any curve. We might for the same purpose, and in the same manner, employ the implicit form (61).

We may arrive at the same result (57) by the consideration of limits, as follows: Let (x,y) be the co-ordinates of the point P (fig. 2.) and (x',y') of the point Q, and through the points Q and P draw the secant line Q R, it is obvious that $y' - y =$ Q L and $x' - x =$ P L, and the two triangles R K V and P Q L being, as before, similar, we have as in (56).

(56a) $$1 : \tan. \mathrm{R} : : , x' - x : y' - y,$$

from which we have

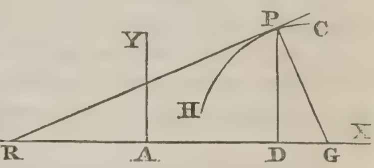
(57a) $$\tan. \mathrm{R} = \frac{y' - y}{x' - x},$$

which is the tangent of the angle which the secant line Q R makes with the axis of x. This angle varies the nearer the point Q approaches to the point P, and in the *limit*, when Q coincides with P the secant line Q R becomes a tangent line P R, the angle at R becomes the angle which the tangent line makes with the axis of x, and $\frac{y' - y}{x' - x}$ becomes $\frac{dy}{dx}$, the ultimate ratio or differential coefficient. Hence, when P R is a tangent line, (57a) becomes (57).

PROPOSITION. III.

Determine the equation of the tangent line to a given plane curve.

FIG. 4.



Let the equation of the plane curve H P be represented by

(62) $$y = \varphi x.$$

Let x' and y' be the variable co-ordinates of the line R P and x and y the co-ordinates of the point of tangency P. The equation of any line R P passing through the point P is,

(63) $$y' - y = a\,(x' - x).$$

Since a in (63) is the tangent of the angle P R X, we have by (57), when P R is a tangent line,

(64) $$\frac{dy}{dx} = a.$$

Put this value of a into (63), and we have

(65) $$y'-y = \frac{dy}{dx}(x'-x),$$

which is the equation of the tangent line P R. This is a general form, being independent of the nature of the curve H P. To apply it to any particular curve, we deduce from (62) the value of the differential coefficient, which being put into (65), gives the tangent line to the particular curve.

As an example, let (62) be for a particular curve,

(c) $$y = ax^3 + b.$$

From this we get

(d) $$\frac{dy}{dx} = 3ax^2.$$

This value of the differential coefficient put into (65,) we have for the tangent line to the curve(c),the equation

(e) $$y'-y = 3ax^2(x'-x).$$

In the general form (65), and in every particular case, it is to be observed that x and y are the co-ordinates of the point of tangency. By means of the equation of the curve (62), we can eliminate y from the tangent line (65).

If we designate the differential of φx, by writing the characteristic d before it, thus, $d.\,\varphi x$, we may write the differential coefficient of (62) in the form

(65a) $$\frac{dy}{dx} = \frac{d.\,\varphi x}{dx},$$

or putting

$$\frac{d.\,\varphi x}{dx} = \varphi' x,$$

we may write (65a)

(65b) $$\frac{dy}{dx} = \varphi' x,$$

and by substituting from (62) and (65b) into (65), we have for the general form of the tangent line to a plane curve,

(65c) $$y'-\varphi x = \varphi' x\,(x'-x).$$

But it is simpler to employ (65).

From (65b) we observe that when the curve (62) is explicit, the differential coefficient is a function of x.

If the equation of the curve be implicit, as in form (61), the process is the same, viz., deduce from (61) the differential coefficient, and put its value into (65).

Ex.—Determine the equation of the tangent line to the curves

$$y = ax^4 + c. \quad \therefore \quad y'-y = 4ax^3\,(x'-x),$$

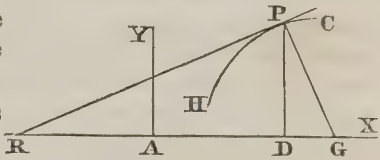
$$y = ax - x^2. \quad \therefore \quad y'-y = (a-2x)\,(x'-x)$$

(C) $$\begin{cases} y = a \log. x, \quad y = a \log. (x+h), \quad y = ax \log. x, \\ x^2 + y^2 - R^2 = o, \quad a^2y^2 + b^2x^2 - a^2b^2 = o, \quad y^2 - px = o. \end{cases}$$

PROPOSITION IV.

Determine the equation of the Normal line to a plane curve.

FIG. 5.



Let PG be the normal line to the curve at P, and (x',y') the co-ordinates of any point on PG, and (x,y) the co-ordinates of the normal point P.

The equation of any line P G through P is of the form

(66) $$y'-y = a\,(x'-x)$$

where a is the tangent of the angle PGX.

If PR be tangent to the curve at P, and (65) its equation, we have by the condition of perpendicular lines, (Analytical Geometry.)

$$\frac{dy}{dx} = -\frac{1}{a},$$

or taking the reciprocal,

$$a = -\frac{dx}{dy}.$$

This value of a put into (66), we have

(67) $$y'-y = -\frac{dx}{dy}\,(x'-x),$$

which is the equation of the normal line P G. If (62) be the equation of the curve H P, we deduce from (62) the differential coefficient and put its reciprocal into (67).

Ex.—Determine the equation of the normal line when the curve is

$$(c) \qquad y = ax^3 + b. \quad \therefore \quad \frac{dy}{dx} = 3ax^2,$$

and the reciprocal of this put into (67), we have

$$(d) \qquad y'-y = -\frac{1}{3ax^2}(x'-x),$$

the normal line required.

Determine the normal line in the curve

$$(e) \qquad x^2 + y^2 - R^2 = o.$$

From this we have

$$\frac{dy}{dx} = -\frac{x}{y},$$

and (67) becomes

$$(f) \qquad y'-y = \frac{y}{x}(x'-x),$$

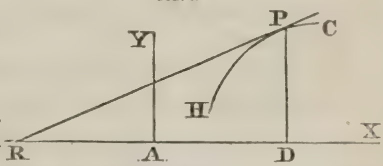
which is the equation of the normal line to the curve (e).

Ex.—Determine the normal line in the curves marked (C).

PROPOSITION V.

Determine the length of the subtangent in a given plane curve.

FIG. 6.



Let the equation of the given curve H P be

$$(68) \qquad y = \varphi x,$$

where x and y are the co-ordinates of the point of tangency P. The subtangent is R D. To find the length of R D we have, by Trigonometry, in the right-angled triangle P R D,

$$(68a) \qquad PD = RD \text{ tan. } R.$$

Put into this the value of tan. R at (57), and we have, since P D is the y of the point P,

(69) $$\text{R D} = y\,\frac{dx}{dy},$$

which is the length of the subtangent required. To apply (69) to any given curve (68), deduce from (68) the differential coefficient, and its reciprocal put into (69), we have the subtangent for the particular curve.

Ex.—Let the curve (68) be

$$y = ax^2 + b. \quad \therefore \quad \frac{dy}{dx} = 2ax, \text{ or } \quad \frac{dx}{dy} = \frac{1}{2ax}.$$

This put into (69), we have

$$\text{R D} = \frac{y}{2ax},$$

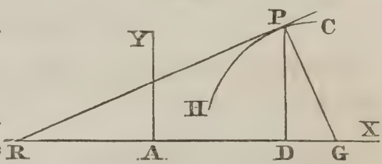
the subtangent required.

Ex.—Determine the subtangent in the curves (C).

PROPOSITION VI.

Determine the length of the subnormal in a given plane curve.

FIG. 7



Let the equation of the given curve H P be

(70) $y = \varphi x.$

Draw P G the normal, then D G is the subnormal whose length is required.

The right-angled triangles R P G and D P G, being similar, we have the angle P R D equal to the angle D P G. Hence by (57)

(71) $$\frac{dy}{dx} = \text{tan. D P G}.$$

From the right-angled triangle P D G we have, by Trigonometry,

(72) $$\text{D G} = \text{P D tan. D P G}.$$

By (71) this becomes, (since P D is the ordinate y),

(73) $$DG = y\frac{dy}{dx} = \text{subnormal}.$$

We might eliminate y from (73) by means of the equation of the curve (70), and the subnormal would be a function of x.

To apply (73) to any given curve, deduce from the equation of the curve (70), the differential co-efficient, and put its value into (73).

Ex.—Determine the subnormal of the curve

(c) $$y = ax^3 + b, \therefore \frac{dy}{dx} = 3ax^2,$$

and (73) becomes

(d) $$DG = 3ax^2\,y, \text{ the subnormal.}$$

If we eliminate one of the co-ordinates, as y from (d) by means of (c), we have for the subnormal as a function of x,

(e) $$DG = 3ax^2\,(ax^3 + b).$$

Ex.—Determine the subnormal in equations (C).

PROPOSITION VII.

Determine the length of the tangent or normal of a given curve.

Let HP [Fig. Proposition VI.] be the given curve, and (70) its equation. PR is the length of the tangent, and PG of the normal.

From the right-angled triangle PRD we have

$$\overline{PR}^2 = \overline{PD}^2 + \overline{RD}^2.$$

Put in this for RD its value at (69), and since PD is the ordinate y, we have for the length of the tangent

(74) $$PR = \left(y^2 + y^2\frac{dx^2}{dy^2}\right)^{\frac{1}{2}},$$

In like manner, for the length of the normal PG we have

$$\overline{PG}^2 = \overline{PD}^2 + \overline{DG}^2.$$

Put in this for DG its value at (73), and we have for the length of the normal

(75) $$PG = \left(y^2 + y^2\frac{dy^2}{dx^2}\right)^{\frac{1}{2}},$$

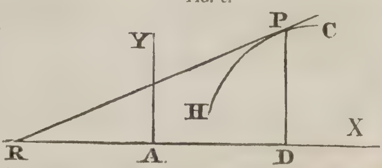
The differential coefficient in (75) and (74) is to be supplied from (70), the equation of the curve.

Ex.—Determine the length of the tangent and normal lines in equations (C.)

PROPOSITION VIII.

Determine the point on a given curve, from which, if a tangent line be drawn, it will make a given angle with the axis of abscissas.

FIG. 8.



Let the equation of the given curve HP be

(76) $y = \varphi x.$

Since the angle PRX, which the tangent line makes with axis of x is given, its tangent is given. Hence putting m for the tangent of PRX, we have by (57)

(77) $$\frac{dy}{dx} = m.$$

The two equations (76) and (77) solved for x and y make known the co-ordinates of P, the point of tangency required.

Ex.—For a particular curve let (76) be

(c) $y = ax^2 + b \therefore \dfrac{dy}{dx} = 2\,ax$ and (77) becomes

(d) $2\,ax = m$, equations(c)and(d)solved for x and y give

(e) $$x = \frac{m}{2\,a}, \quad y = \frac{m^2}{4a} + b$$

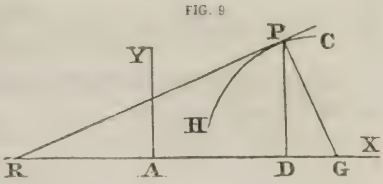
which are the co-ordinates of P the point required.

Ex.—Determine such a point of tangency in equations (C).

PROPOSITION IX.

Determine the point on a plane curve from which if a tangent and normal be drawn, the subnormal will be a given multiple of the subtangent.

FIG. 9



Let the equation of the given curve HP be

(78) $y = \varphi x.$

Let m be the multiple that DG is of RD, then by the proposition

(79) $$DG = m.\ DR$$

substitute into this for DG and DR, their values in (73) and (69). and we have

(80) $$y\,\frac{dy}{dx} = m\ y\,\frac{dx}{dy}$$

from which we have

(81) $$\frac{dy}{dx} = \pm\ \sqrt{m}.$$

Equations (78) and (81) are sufficient to determine x and y, the co-ordinates of the point required.

Ex.—For a particular curve let (78) be

(c) $y = ax^2 + b.$ From this we get

$$\frac{dy}{dx} = 2\,ax,$$

and (81) becomes

(d) $$2\,ax = \pm\ \sqrt{m},$$

solve equations (c) and (d) for x and y and we have the co-ordinates of the point required.

Ex.—Determine such a point in curves (C).

PROPOSITION X.

Determine the point on a curve from which if a tangent line be drawn the subtangent will be of a given length.

Let the equation of the given curve HP (fig. 9.) be

(82) $$y = \varphi x,$$

putting n for RD the given subtangent, (69) gives

(83) $$n = y\,\frac{dx}{dy},$$

after putting into (83) the value of $\frac{dx}{dy}$ deduced from (82), equations (82) and (83) solved for x and y make known the point required.

Ex.—Determine such a point in curves (C).

PROPOSITION XI.

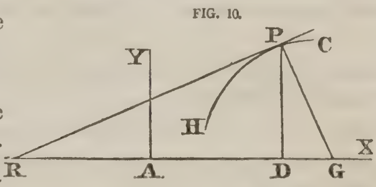
Determine the point on a curve from which if a tangent line be drawn it will be of a given length.

Proposition X. suggests the method of doing this.

PROPOSITION XII.

Determine the point on a curve from which if a tangent and normal be drawn they will intercept a given distance on the axis of abscissas.

FIG. 10.



Let the given curve HP be represented by

(84) $$y = \varphi x.$$

It is obvious that RG the part of the axis of abscissas intercepted by the tangent PR and the normal PG is composed of the subtangent and subnormal. Hence, putting c for the given intercept RG we have by means of (69) and (73)

(85) $$c = y\frac{dx}{dy} + y\frac{dy}{dx},$$

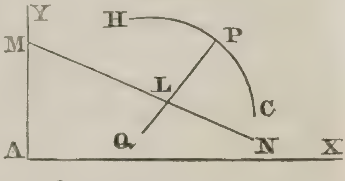
Having put into (85) for the differential coefficients, their values deduced from (84), the equations (84) and (85) make known the coordinates x and y of the point required.

Ex.—Determine such a point in curves (C).

PROPOSITION XIII.

Determine the distance from a given point on a curve to a given line, measured on the normal to the curve.

FIG. 11.



Let (x, y) be the given co-ordinates of the point P, and let the equation of the curve HC be

(86) $\qquad y = \varphi x,$

and the equation of the given line MN be

(87) $$y' = ax' + b.$$

The equation of the normal PQ is by (67′)

(88) $$y' - y = -\frac{1}{p}(x' - x)$$

where for brevity we put

$$p = \frac{dy}{dx},$$

and where x', y' are the variable co-ordinates of PQ.

If x and y be the co-ordinates of P, and x' and y' of L, the distance PL, which call N, is by a well known form in Analytical Geometry,

(89) $$N^2 = (y' - y)^2 + (x' - x)^2.$$

At the point L, x' and y' are common to (89), (87) and (88). Find the values of x' and y' from (87) and (88) and substituting them into (89) we have for the length required,

(90) $$N = \frac{y - ax + b}{ap + 1}(1 + p^2)^{\frac{1}{2}}.$$

The differential coefficient p deduced from (86) and its value put into (90), we have the distance PL in terms of the given co-ordinates of P and other known quantities.

Determine such a distance for the following curves.

(D) $\quad y = ax^2, \quad x^2 + y^2 - R^2 = o, \quad a^2 y^2 + b^2 x^2 - a^2 b^2 = o.$

PROPOSITION XIV.

Determine the point on a plane curve from which if a normal line be drawn the part of it intercepted between the curve and a given straight line will be of a given length.

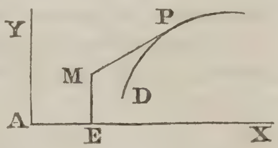
This is obviously effected by solving (86) and (90) for x and y, the co-ordinates of the point P. Hence, putting for the differential coefficient p in (90) its value from (86), we have (86) and (90) to find x and y in terms of the given line N and other known quantities.

Propositions similar to XIII. and XIV, might also be solved in relation to the tangent line. The method of the solution is readily suggested by these Propositions.

PROPOSITION XV.

From a given point a tangent line is drawn to a given curve, determine the co-ordinates of the point of tangency.

FIG. 12.



Let M be the given point, and (m, n) its co-ordinates.

Let the equation of the given curve PD be

(91) $$\varphi(x, y) = o.$$

Let MP be the tangent line. From (65) the equation of MP is

(92) $$y' - y = p\,(x' - x)$$

where p is put for the differential coefficient, an abbreviation we will frequently employ hereafter, and x' and y' are the variable co-ordinates of MP.

At the point M, x' and y' become m and n, and (92) becomes

(93) $$n - y = p\,(m - x).$$

For the differential coefficient p in (93) put its value taken from (91), and then solving (91) and (93) for x and y we have the co-ordinates of the point P required.

Ex. 1. Let (91) be the circle,

(a) $$x^2 + y^2 - \mathrm{R}^2 = o.$$

The differential coefficient from this is

$$(b) \qquad \frac{dy}{dx} = -\frac{x}{y} = p,$$

and (93) becomes

$$(c) \qquad n - y = -\frac{x}{y}(m - x),$$

solve (a) and (c) for x and y, and we have the point P required.

Ex. 2. Determine such a point of tangency in the curves (D.)

It is obvious that if (91) be a curve of the n[th] degree, there will be n points of tangency, all of which will be made known by the n values of x and y found by solving (91) and (93). Thus, example (a) is of the 2[d] degree, and the solution of equations (a) and (c) gives two values for x and two for y, which shows, that from the same point M two tangent lines can be drawn to the circle (a).

PROPOSITION XVI.

From a given point a normal is drawn to a given curve, determine the co-ordinates of the normal point.

By the normal point, a term already used, is understood the point on the curve where the normal intersects it.

From (67) we have for the equation of the normal line

$$(93a) \qquad y' - y = -\frac{1}{p}(x' - x).$$

Proceed with this equation as with (92), and the solution is the same as in Proposition XV.

Ex.—Find such a point in the curve

$$(b) \qquad y = ax^2.$$

From this, $p = 2ax$, and (93a) becomes, (putting n and m for y' and x')

$$(c) \qquad n - y = -\frac{1}{2ax}(m - x).$$

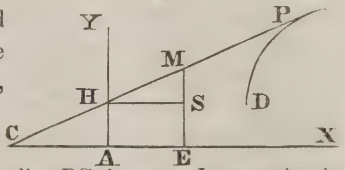
Proceeding with the solution of (b) and (c), we find a cubic equation which shows that from any point not on the curve, three normals can be drawn to the curve (b).

Ex.—Determine such a point in the curves (D).

PROPOSITION XVII.

Determine the distance from a given point to each of the co-ordinate axes, measured on the tangent to a given curve.

FIG. 13.



Let M be the given point, and (m,n) its co-ordinates. Let the equation of PD, the given curve, be represented by

(94) $$\varphi(x,y) = o.$$

Through M suppose the tangent line PC drawn. Its equation is, as in (93), for the point M,

(95) $$n-y = p\,(m-x).$$

The general equation of the same line is, as in (92),

(96) $$y'-y = p\,(x'-x).$$

To get the distances AH and AC, put first in (96) y' zero, and we have for x',

(97) $$x' = x - \frac{y}{p} = \text{AC}.$$

Again, put in (96) x' zero, and we have for y',

(98) $$y' = y - p\,x = \text{AH}.$$

Equations (97) and (98) make known AC and AH in terms of x,y, and constants, (x and y being the co-ordinates of the point of tangency P.)

Having also the co-ordinates (m,n) of the point M, it is evident that the required distances MH and MC are readily expressed from the right-angled triangles CEM and HSM, in terms of AH, AC, AE, and EM.

These expressions for MH and MC will involve the co-ordinates x and y of the point of tangency, which co-ordinates may be determined by solving (94) and (95) for x and y.

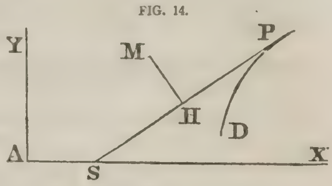
Determine these distances in the curves (D).

If the proposition were to determine similar distances measured on the normal line to a given curve, the solution would be effected in the same manner, by taking the equation of the normal line instead of (95) and (96).

PROPOSITION XVIII.

Determine the distance from a given point to the tangent of a given curve.

FIG. 14.



Let PD be the given curve, and

(99) $\varphi(x,y) = o,$

its equation.

Let (m,n) be the given point M, and x' and y' being the variable co-ordinates of PS, the equation of the tangent line PS is

(100) $$y'-y = p\,(x'-x).$$

From M draw MH perpendicular to PS, and MH is the distance required. For the equation of the line MH, passing through M and perpendicular to PS, we have

(101) $$y'-n = -\frac{1}{p}\,(x'-m).$$

For the required distance MH we have, by Analytical Geometry,

(102) $$\overline{\mathrm{MH}}^2 = (y'-n)^2 + (x'-m)^2,$$

where x' and y' are the co-ordinates of the point H.

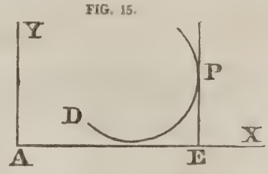
At the point H, x' and y' are common to (100), (101) and (102), hence solve (100) and (101) for x' and y', and putting their values into (102), the required distance MH becomes known. This distance is expressed in terms of constants, and of the co-ordinates x and y of the point of tangency. These co-ordinates are given by the proposition.

Determine this distance in the curves (D).

PROPOSITION. XIX.

Determine the point on a curve at which the tangent line is perpendicular to the axis of abscissas.

FIG. 15.



Let PD be the given curve, and

(103) $\varphi(x,y) = o,$

its equation.

Let P be the point at which the tangent line PE is perpendicular to the axis of abscissas.

Since $\frac{dy}{dx}$ expresses the tangent of the angle which the tangent lines makes within the axis of abscissas, we have, when this is a right angle,

$$\frac{dy}{dx} = \text{Infinity},$$

or since the reciprocal of infinity is zero,

(104) $$\frac{dx}{dy} = o.$$

Equations (103) and (104) serve to determine x and y, the co-ordinates of the point required.

The same result would be obtained by considering that the co-tangent of an angle is the reciprocal of the tangent, and that when the angle E is a right angle, its co-tangent is zero, which gives (104).

Ex.—Determine such a point when the curve PD is the circle.

(*b*) $$(x-a)^2 + (y-\beta)^2 - R^2 = o.$$

Differentiate this, and we have

(*c*) $$2\,(x-a)\,dx + 2\,(y-\beta)\,dy = o.$$

From this we get

(*d*) $$\frac{dx}{dy} = -\frac{y-\beta}{x-a}.$$

By (104) this is zero. Hence

(*e*) $$-\frac{y-\beta}{x-a} = o.$$

Solve (*e*) and (*b*) for x and y, and we have

(*f*) $$y = \beta, \qquad x = a \pm R,$$

which shows that there are two points on the circle (*b*) at which the tangent line is perpendicular to the axis of abscissas.

Ex.—Determine such a point in the curves

(E) $y = ax^2 + b, \qquad (y-\beta)^2 - nx = o, \qquad y^2 = mx.$

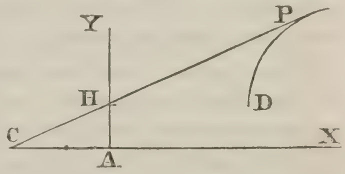
PROPOSITION XX.

Determine when a curve has a rectilinear asymptote, and construct the asymptote.

Definition.—A rectilinear asymptote is a tangent line whose point of tangency is infinitely distant.

Such a line consequently approaches continually nearer to a curve without, however, touching it at any finite distance.

FIG. 16.



Let PD be a given curve, and

(105) $y = \varphi x$

its equation. The equation of a line tangent to this curve is of the form

(106) $$y' - y = p\,(x' - x),$$

where x and y are the co-ordinates of the point of tangency. In (106) put $y' = o$ and we have

(107) $$x' = x - \frac{y}{p} = \text{AC}.$$

Again in (106) put $x' = o$, and we have

(108) $$y' = y - px = \text{AH}.$$

The differential coefficient p derived from (105) obviously does not contain y. It may therefore be regarded as a function of x. Let us then for brevity put

(109) $$p = \Psi x,$$

and substitute (105) and (109) into (107) and (108), and we have

(110) $$\text{AC} = x - \frac{\varphi x}{\Psi x},$$

and

(111) $$\text{AH} = \varphi x - x\Psi x.$$

In order to determine whether PD can have an asymptote, we make x infinite in (110) and (111), and observe what AC and AH

then become. If both the values of AC and AH become infinite when x is infinite, the curve has no asymptote, for no line can then be drawn which will become a tangent at a point infinitely distant.

If either or both of the values of AC and AH become finite or zero when x is infinite, the curve has an asymptote, because a line drawn in a definite manner becomes a tangent at a point infinitely distant.

If both the values of AC and AH are definite lines when x is infinite, the line drawn through the points C and H is the asymptote.

If both these values are zero for x infinite, the asymptote passes through the origin A, and to draw it, we must know the angle it makes with one of the co-ordinate axes.

The tangent of the angle which the asymptote makes with the axis of x is found by making x infinite in (109).

If for x infinite one of the intercepts (110), (111) is infinite, and the other finite, the curve has an asymptote parallel to the axis on which the intercept is infinite. If one of the intercepts (110,) (111) be infinite, and the other zero, the axis is the asymptote.

If for any given value of x in (110), (111), one of the intercepts AC, AH becomes infinite, and the other finite or zero, that value of x is the abscissa of a point which is infinitely distant, and the curve has an asymptote parallel to the axis of y, if x be an actual value, and coinciding with the axis of y, if x be zero.

We will illustrate this by an example.

Take as a particular case of (105) the equation

$$(c) \qquad y = \frac{b}{c}(x^2 - a^2)^{\frac{1}{2}},$$

which is the equation of the hyperbola.

Proceed with the equation of its tangent line as above directed, and we find the intercepts corresponding to (110), (111), each zero when x is infinite. Hence the hyperbola (c) has an asymptote passing through the origin.

To draw this asymptote we must find the angle it makes with the axis of x. For this purpose, the differential coefficient deduced from (c) may be put into the form

(d) $$p = \frac{b}{a}\left(1 - \frac{a^2}{x^2}\right)^{-\frac{1}{2}}$$

which corresponds to (109). When x is infinite, (d) becomes

(e) $$p = b \div a,$$

which makes known the angle that the asymptote makes with the axis of x; and the asymptote, passing also through the origin, can be drawn.

Ex. 2.—Determine whether the curve

(f) $$y = \frac{a^2}{x},$$

has an asymptote. Differentiate this, and we have

(g) $$\frac{dy}{dx} = -\frac{a^2}{x^2}.$$

This is the value of ψx in (109). Hence substituting from (f) and (g) into (110) and (111), we have

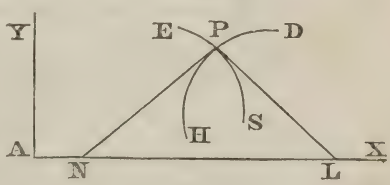
(h) $$\mathrm{AC} = 2x, \text{ and } \qquad \mathrm{AH} = \frac{2a^2}{x}.$$

If in the values (h) x be made infinite, we have AC infinite, and AH zero. Hence the axis of x is an asymptote. If in (h) x be zero, AC is zero and AH is infinite, from which we infer that the axis of y is also an asymptote.

PROPOSITION XXI.

Two curves intersect, determine the angle they make with each other at the point of intersection.

FIG. 17.



Let HD and ES be the intersecting curves, intersecting at the point P.

Let the equation of ES be represented by

(112) $\phi(x,y) = o$,

and the equation of HD by

(113) $\psi(x,y) = o$.

The angle made by two intersecting curves is the same as the angle made by their tangents at the point of intersection. At the point of intersection P, draw PN and PL tangent to the curves HD and ES respectively. Let

p = differential coefficient of (112) = tan. PLX, and
p' = differential coefficient of (113) = tan. PNX.

If V = tangent of the angle NPL, then since NPL is the difference of the angles PLX and PNX, we have, by a well known Theorem in Trigonometry, for the tangent of the difference of two angles,

(114) $$V = \frac{p - p'}{1 + pp'},$$

This is the value of the required angle in terms of the differential coefficients p and p'. As p and p' are functions of x and y, if the values of p and p' be deduced from (112) and (113), and substituted into (114), V will then be a function of x and y, and we may represent (114) by

(114a) $$V = F\,(x,y),$$

where $F\,(x,y)$ is put for what (114) becomes when the values of p and p' are substituted into it. Now solve (112) and (113) for x and y, the co-ordinates of the point of intersection, and their values put into (114a), we have the tangent of the required angle in known terms.

Ex.—Find the angle of intersection of the circle and parabola.

Here (112) and (113) become

(c) $$x^2 + y^2 - R^2 = o, \text{ and } \qquad y^2 - 4mx = o,$$

from which we get respectively,

(d) $$\frac{dy}{dx} = -\frac{x}{y} = p, \text{ and } \qquad \frac{dy}{dx} = \frac{2m}{y} = p'.$$

The values of p and p' in (d) put into (114), we have for (114a),

(e) $$V = \frac{y\,(x + 2m)}{2mx - 1}.$$

Solve equations (c) for x and y, and putting their values into (e), we have the value of V in known terms.

Ex.—Find the angle of intersection of the two curves,

$$y^2 - 4mx = o, \qquad y^2 + (x - c)^2 - R^2 = o,$$

or of the two curves,

$$y = ax^2, \qquad (y - \beta)^2 - mx = o.$$

PROPOSITION XXII.

Determine the relation between the parameters of two curves tangent to each other.

The constants that enter into the equation of a curve are called Parameters. Thus, if we have the equations,

$$(F) \qquad \begin{cases} y^2 - 4\,m\,x = o, \\ y^2 + x^2 - R^2 = o, \\ (y - \beta)^2 + (x - a)^2 - R^2 = o, \end{cases}$$

m is a parameter in the first, R is a parameter in the second, and a, β, R are parameters in the third.

We may designate in general that the equation of a curve contains any particular parameter, as R, by writing the equation thus,

$$(115) \qquad \phi(x, y, R) = o,$$

which signifies that the equation of the curve contains the variable co-ordinates, and the particular parameter R. Beside the particular parameter expressed, the equation may contain others not appearing in the general form of the equation. Thus (115) may represent either the second or third of equations (F).

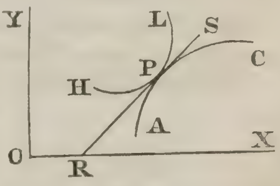
This premised, we proceed to the proposition.

Let HL be one curve, and

$$(116) \qquad \phi(x, y, m) = o$$

its equation, in which m is a parameter.

FIG. 18.



Let AC be another curve, and

$$(117) \qquad F\,(x, y, n) = o,$$

its equation, in which n is a parameter.

Suppose these curves tangent at the point P. A line SR, tangent to AC at P, will also be tangent to HL at the same point.

If p and p' represent the differential coefficients of (116), and

(117), for the common point P, then since (by Proposition II.) p and p' each represent the tangent of the same angle R, we have

(118) $$p = p',$$

in which it is to be recollected that p and p' are functions of x and y.

As p and p' are functions of the co-ordinates of the point of tangency these co-ordinates x and y may be eliminated between (116), (117), and (118). The resulting equation solved for m may be represented by

(119) $$m = f.\, n,$$

which is the required relation between the parameters m and n.

Ex.—Let the curves be the two circles,

(a) $$x^2 + y^2 = n^2,$$

(b) $$(x - a^2)^2 + (y - \beta)^2 = m^2,$$

Equating the values of the differential coefficients of (a) and (b) we have, after clearing of fractions,

(c) $$ay = \beta x.$$

Eliminate x and y from (a), (b), and (c), and we have for the relation between m and n,

(d) $$m = (a^2 + \beta^2)^{\frac{1}{2}} \pm n.$$

Equation (d) shows, that with the same centre there are two radii with which the circle (b) may be described so as to be tangent to the circle (a), one giving internal, the other external contact.

From (d) we have

(e) $$(a^2 + \beta^2)^{\frac{1}{2}} = m \mp n,$$

which shows, (since $(a^2 + \beta^2)^{\frac{1}{2}}$ is obviously the distance of the centres) that when two circles have external or internal contact, the distance of their centres equals the sum or difference of their radii,—a proposition in plane Geometry.

Determine the relation between the parameters m and n in the lines,

$$\begin{cases} y = ax + m \\ x^2 + y^2 = n^2, \end{cases} \quad \text{or} \quad \begin{cases} y = ax + m \\ n^2 y^2 + c^2 x^2 = n^2 c^2, \end{cases}$$

when they are tangent to each other.

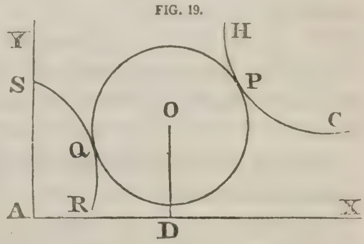
Equation (118) may be regarded as the condition of the tangency

of two curves. This condition results from the obvious principle, that if two curves are tangent to each other, they have a common tangent line at the point of contact.

PROPOSITION XXIII.

A circle touches two given curves, determine the Locus of its centre.

FIG. 19.



Let HC be one of the curves, and

(120) $\phi(x', y') = o$,

its equation.

Let SR be the other curve and

(121) $F(x'', y'') = o$,

its equation. Let

(122) $$(x - a)^2 + (y - \beta)^2 = R^2,$$

be the equation of the circle tangent to (120) and (121) at P and Q, and in which a and β are the co-ordinates of the centre O. At P the co-ordinates of (120), and (122) are common. For that point (122) may be written,

(123) $$(x' - a)^2 + (y' - \beta)^2 = R^2.$$

At (Q) the co-ordinates are common to (121) and (122). Hence, for that point (122) may be written,

(124) $$(x'' - a)^2 + (y'' - \beta)^2 = R^2.$$

Let p = the differential coefficient of (120) for the point P,

$p' =$	do.	(123)	P,
$P =$	do.	(121)	Q,
$P' =$	do.	(124)	Q.

Since (120) and (123) are tangent at P, we have, as in (118),

(125) $$p = p'.$$

For a similar reason we have,

(126) $$P = P'.$$

In these equations p and p' are functions of x' and y' and P and P′ are functions of x'' and y''.

The co-ordinates of the point P are common to (120), (123) and (125). Eliminating these co-ordinates between these three equations, the resulting equation will contain a, β, and R, and may be represented by

(127) $$\phi(a, \beta, R) = o.$$

The co-ordinates of the point Q are common to (121), (124), and (126). Eliminating these co-ordinates between these three equations the resulting equation will contain a, β, and R, and may be represented by

(128) $$F(a, \beta, R) = o.$$

Eliminate the variable radius R between (127), and (128). The resulting equation will contain a and β, and may be represented by

(129) $$\beta = \psi a.$$

This being an equation between the co-ordinates of the centre, is the curve required.

Ex.—Let the two given curves, HC and SR, be a circle, and straight line. Then (120) and (121) become

(c) $$x'^2 + y'^2 - m^2 = o, \text{ and } \quad y'' - nx'' - b = o,$$

the differential coefficients from which are

(d) $$\frac{dy'}{dx'} = -\frac{x'}{y'} = p, \text{ and } \quad \frac{dy''}{dx''} = n = P.$$

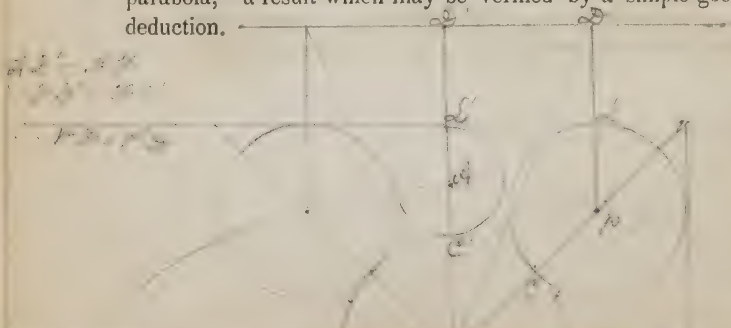
The differential coefficients of (123) and (124) are

(e) $$\frac{dy'}{dx'} = -\frac{x'-a}{y'-\beta} = p', \text{ and } \frac{dy''}{dx''} = -\frac{x''-a}{y''-\beta} = P',$$

and equations (125) and (126) become respectively,

(f) $$\frac{x'}{y'} = \frac{x'-a}{y'-\beta}, \text{ and } \quad n = -\frac{x''-a}{y''-\beta}.$$

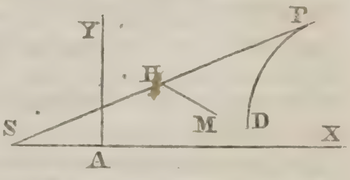
Eliminate x' and y' between (123) and the first of (c) and (f), and we have the particular form corresponding to (127). Eliminate x'' and y'' between (124) and the second of (c) and (f), and we have the particular form corresponding to (128). Eliminate R between these two results, and we have for the locus of the centre a parabola,—a result which may be verified by a simple geometrical deduction.



PROPOSITION XXIV.

From a given point a line is drawn making a given angle with the tangent of a curve, determine the locus of the intersection.

FIG. 20.



Let PD be the given curve, and

(130) $\qquad \phi(x,y) = o,$

its equation.

Let M be the given point, and (c,d) its co-ordinates.

Let PS be the tangent line, and MH the line drawn through M, making at H the given angle with the tangent PS.

Let V = tangent of the given angle at H, and let (m,n) be the co-ordinates of H,—the point of intersection.

The equation of the tangent line PS for the point H is, as in (93),

(131) $$n-y = p\,(m-x).$$

The equation of MH passing through M is

(132) $$y'-d = \pi(x'-c).$$

For the point H this becomes,

(132a) $$n-d = \pi(m-c).$$

In these equations π is the tangent of the angle which HM makes with the axis of x.

For the tangent of the angle H we have, as in (114),

(133) $$\mathrm{V} = \frac{p-\pi}{1+p\,\pi},$$

in which p is a function of x and y.

Eliminate x,y, and π between the four equations (130), (131), (132a), and (133).

The resulting equation will contain no other variables than m and n. Representing it by

(134) $$\psi(m,n) = o,$$

we have the locus required.

Cor. 1st.—If the angle H be a right angle, instead of (133) we have

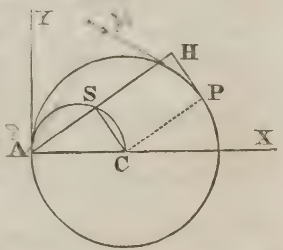
(135) $$1 + p\pi = o,$$
and x,y, and π are then to be eliminated between (130), (131), (132a), and (135).

Cor. 2d.—If the tangent of the angle H, instead of being constant, be a given function of the co-ordinates of the point H, i. e. if
$$\mathrm{V} = \phi(m,n),$$
then instead of (133) we have

(136) $$\phi(m,n) = \frac{p-\pi}{1 + p\pi}$$
and x, y and π being eliminated between (130), (131), (132a), and (136), we have the locus required.

Ex. 1.—Let the curve be a circle, and the point from which lines perpendicular to the tangent are drawn, be on the perimeter, find the locus of the intersection.

FIG. 21.



Take the origin A at the point through which the perpendiculars are drawn.

Here the curve (130) is

(a) $$y^2 = 2rx - x^2.$$

The tangent line HP is, as in (131),

(b) $$n - y = p\,(m - x).$$

The equation of the perpendicular AH, passing through the origin and the point H, whose co-ordinates are m and n, is for the point H,

(c) $$n = \pi m,$$
and H being a right angle, we have, as in (135),

(d) $$1 + p\pi = o.$$

Eliminate x,y, and π from (a), (b), (c), and (d), by the ordinary rules of Algebra, and the resulting equation is

(e) $$n^2\, r^2 = (m^2 + n^2)^2 - 2rm\,(m^2 + n^2),$$
the locus of the point H. This locus might be immediately obtained from the figure, by joining C and P, then CP is parallel and equal to SH.

If (e) be changed to a polar equation, A being the pole, AH the radius vector R, and AX the angular axis, we have

(136f) $$R = r \cos. \omega + r,$$

a very simple result, and which shows that if a semicircle be described on the radius AC, and any line AS be drawn through A, and produced till SH equals the radius AC, a line through H perpendicular to AH is tangent to the circle. As the curve (136f) possesses many remarkable properties, we will refer to it hereafter by the name of, THE CIRCLOID.

Ex. 2.—Find the locus when the curve is an ellipse or hyperbola, and the point through which the lines are drawn is the focus, or centre, or vertex, the angle H being right.

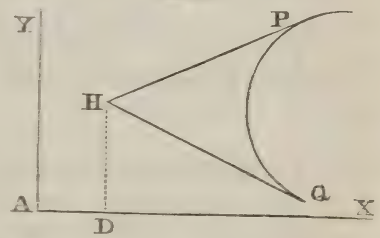
Ex. 3.—Find the locus when the curve is the parabola, and the point through which the lines are drawn is the focus or vertex, and the angle of intersection H is either right, or oblique.

The properties of these loci are more readily investigated by changing their rectangular to polar equations, as was done with (e).

PROPOSITION XXV.

Pairs of tangents to a curve intersect at a given angle, determine the locus of their intersection.

FIG. 22.



Let PQ be the given curve, and PH and QH a pair of tangents to it intersecting at H.

Let (x',y') be the co-ordinates of P, one of the points of tangency, and (x'',y'') the co-ordinates of Q, the other point of tangency.

Let (m,n) be the co-ordinates of H, and let the equation of PQ for the point P be,

(137) $$\phi(x',y') = o,$$

and the equation of PQ for the point Q,

(138) $$\phi(x'',y'') = o.$$

Let p and p' be the differential coefficients of (137), and (138).

The equation of the tangent line PH for the point H is as in (93),

(139) $$n-y' = p\,(m-x').$$

The equation of QH for the point H is

(140) $$n-y'' = p'\,(m-x'').$$

Put V = tangent of the given angle H, then as in (114), we have for the tangent of the angle made by the lines, (139), and (140),

(141) $$V = \frac{p-p'}{1+pp'},$$

Since p is a function of x',y', and p' of x'',y'', we have the five equations (137), (138), (139), (140), and (141), to eliminate the four co-ordinates of the points of tangency. The resulting equation will contain no other variables than m and n, and may be represented by

(142) $$\psi(m,n) = o,$$

which is the equation of the locus required.

Note.—If the curve (137) be of the second degree, the two values of each co-ordinate given, by the solution of (137) and (139), make known the co-ordinates of both points of tangency, and (138) and (140) are superfluous. This is indeed true, whatever be the degree of equation (137), for if (137) be of the n^{th} degree, then as was remarked at the close of Prop. XV., the points of tangency are all given by the solution of equations (137) and (139).

Cor. 1st.—If the angle H of intersection be a right angle, we have,

(143) $$1 + pp' = o,$$

which is to be used in this case instead of (141).

Cor. 2d.—If the tangent of the angle H, instead of being constant, were a given function of the co-ordinates of the point H, i. e. if $V = \phi(m, n)$, we have,

(144) $$\phi(m, n) = \frac{p - p'}{1 + pp'},$$

which is to be used in this case instead of (141).

Ex. 1.—Let the curve PQ be a parabola, what is the Locus of the intersections of pairs of tangents?

$$(a) \qquad V^2 = \frac{4n^2 - 8rm}{(2m + r)^2}$$

is the equation of the Locus determined as above directed, which is a hyperbola. If the angle be a right angle then (a) becomes

$$(b) \qquad 2m + r = o,$$

which is a straight line—to wit, the directrix.

Ex. 2.—Let the curve be the ellipse or hyperbola, and the angle H a right angle, what is the Locus of H?

Ans. A circle.

Ex. 3d.—Let the curve PQ be a parabola, and let the tangent of the angle H vary as a given function of the co-ordinates of H, what is the Locus.

Representing the given function of the co-ordinates of H by $\varphi(m, n)$, we have, $V = \varphi(m, n)$.

If we put this value of V into (a), we have

$$(d) \qquad \varphi(m, n) = \frac{2\,(n^2 - 2rm)^{\frac{1}{2}}}{2m + r},$$

which is the locus required. By giving any definite value to $\varphi(m, n)$, we may construct the curve (d).

PROPOSITION XXVI.

Pairs of tangents to a curve make angles with the axis of abcissas, whose product, or sum, or difference is constant, determine the locus of their intersections.

The figure and solution is the same as in last Proposition, except that, instead of (141) we have in case of the product

$$(145) \qquad pp' = c,$$

a constant, and in case of the sum, we have

$$(146) \qquad p + p' = c,$$

and in case of the difference, we have

$$(147) \qquad p - p' = c.$$

Eliminate the co-ordinates of the points of tangency by means of equations (137), (139), and (145), and we have the locus when the product of the tangents is given. For the locus, when the sum

of the tangents is given, we have (137), (139), and (146), to eliminate the co-ordinates of the points of tangency, and when the difference is given we have (137), (139), and (147).

Note 1*st*.—This proposition might have been more simply stated in the general form, thus:

Pairs of tangents to a curve make angles with the axis of abscissas, any function of whose tangents is constant, determine the locus of the intersections.

Designating the given function of the tangents by $\varphi(p, p')$, we have by the proposition,

(148) $$\varphi(p, p') = c.$$

This equation embraces the particular cases (145), (146), (147), and indeed, (141), as also every possible combination of the tangents p and p'.

Note 2*d*.—If the given function of the tangents, instead of being constant, varied as a given function of either or both of the co-ordinates of the point of intersection, we would have, when the function of the tangents varied as as a function of the abscissa,

(149) $$\varphi(p, p') = \Psi\, m,$$

and when it varies as a function of the ordinate.

(150) $$\varphi(p,p') = \psi n,$$

and when it varies as a function of both co-ordinates,

(151) $$\varphi(p,p') = \psi(m,n).$$

We would then have (137), (139), and one of the three last equations to eliminate the co-ordinates of the points of tangency.

Ex. 1.—Pairs of tangents to a parabola make angles with the axis of abscissas, the sum of whose tangents varies as the abscissa of the point of intersection, find the locus of their intersection.

Here (149) becomes

(a) $$p + p' = cm,$$

where c is any constant, and (137), and (139) become

(b) $$y^2 - 4rx = o, \text{ and } \qquad yn - 2r\,(m + x) = o.$$

Eliminate x and y from the three equations (a) and (b), and we have for the locus required,

(c) $$m^2c = n, \text{ a parabola.}$$

Ex. 2.—Pairs of tangents to a parabola make angles with the

axis of abscissas, the difference of whose tangents is constant, find the locus of intersection.

Ans. A hyperbola.

Ex. 3.—Pairs of tangents to a parabola make angles with the axis of abscissas, the sum of whose tangents varies as the ordinate of the point of intersection. Find the locus of intersection.

Here (150) becomes

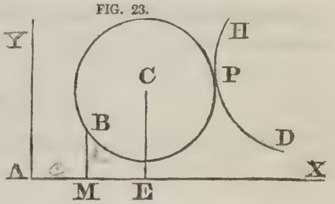
$$p + p' = cn,$$

and the locus is found to be a hyperbola referred to its asymptotes.

PROPOSITION XXVII.

A circle passes through a given point, and touches a given curve, determine the locus of its centre.

FIG. 23.



Let HD be the curve which is touched by the circle at P.

Let B be the given point through which the circle passes, and (c,d) the co-ordinates of B. Then we have

(152) $$(x-a)^2 + (y-\beta)^2 = R^2,$$

for the equation of the circle in any position in which a and β are the co-ordinates of the centre, C. Let

(153) $$\varphi(x,y) = o,$$

be the equation of HD.

At the given point B, whose co-ordinates are (c,d), the circle (152) becomes

(154) $$(c-a)^2 + (d-\beta)^2 = R^2.$$

Let us put p and p' for the differential coefficients of (152) and (153), respectively. At the point P we have, as at (118),

(155) $$p = p',$$

where p and p' are functions of x and y.

At the point P, x and y are common to (152), (153) and (155). Eliminate these co-ordinates between these three equations. The resulting equation will contain a,β, and R. Represent it by

(156) $$F(a,\beta, R) = o.$$

Eliminate R between (154), and (156), the result is an equation of the form

$$(157) \qquad \phi(\alpha,\beta) = o,$$

which is the equation of the curve required.

Ex.—Let the curve HD be the circle,

$$(a) \qquad x^2 + y^2 = r^2,$$

and let the given point B be at M, on the axis of abscissas.

Equating the differential coefficients of (152) and (a), we have

$$(b) \qquad \alpha y = \beta x.$$

Eliminate x,y, and R between the equations (152), (154), (a) and (b). The resulting equation is, since d in (154) is, in this case, zero,

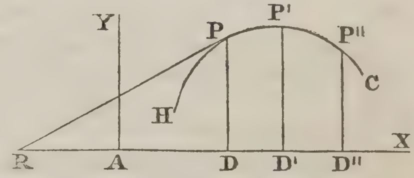
$$(d) \qquad a^2\left(1 - \frac{c^2}{r^2}\right) + \beta^2 - \frac{ac}{r^2}(r^2 - c^2) - \frac{(r^2 - c^2)^2}{4r^2} = o,$$

which is a circle when c is zero, an ellipse when c is less than r, a hyperbola when c is greater than r, and a point, to wit, the centre, when $c = r$.

PROPOSITION XXVIII.

Determine the maximum or minimum ordinate in a plane curve.

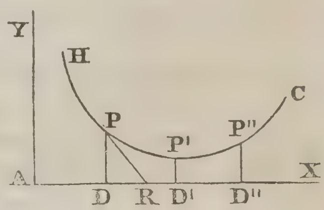
FIG. 24.



Definitions.

1.—A maximum ordinate is greater than the ordinates immediately on each side of it. Thus P′D′ being greater than either of the ordinates, PD, or P″D″, is a maximum ordinate. PD and P″D″ are supposed to be indefinitely near to P′D′.

FIG. 25.



2.—A minimum ordinate is one that is less than either of the ordinates immediately on each side.

Thus P′D′ being less than either PD or P″D″, is a minimum ordinate, PD and P″D″ being indefinitely near to P′D′.

To determine the maximum or minimum ordinate, draw PR tangent to the curve at P. Then by (57)

(158) $$\frac{dy}{dx} = \text{tan. PRX.}$$

Let the equation of the curve HC be,

(159) $$y = \phi x.$$

Suppose P′ be the vertex of the maximum or minimum ordinate.

The nearer the point of tangency P is to P′, the further the angular point R is from the origin. When P′ becomes the point of tangency, the tangent line PR is parallel to the axis of abscissas, the angular point R is infinitely distant, and the angle P′RX is zero, or 180°. But when an angle is zero, or 180°, its tangent is zero; consequently when P′ is the point of tangency, (158) becomes

(160) $$\frac{dy}{dx} = o.$$

The two equations (159), and (160), suffice to determine x and y, the co-ordinates of P′.

Ex. 1.—Let the curve HC be,

(a) $$y = ax - x^2.$$

Then

(b) $$\frac{dy}{dx} = a - 2x.$$

By (160) this is zero, i. e.

(c) $$a - 2x = o.$$

Solve (a) and (c) for x and y, and we have,

(d) $$x = \frac{a}{2}, \text{ and } \qquad y = \frac{a^2}{2}.$$

These are the co-ordinates of the point whose ordinate is a maximum, and equation (d) shows the value of this maximum ordinate, or the co-ordinates of the point on the curve, whose ordinate is a maximum.

This Proposition is merely a particular case of Proposition VIII. It occupies, however, a large space in most books on the Calculus, and is worthy of particular attention.

Determine the maximum ordinate in the following curves.

$$(G)\qquad \begin{cases} y = ax - bx^2, \\ y = 2x^2 - ax^3, \end{cases} \qquad \begin{matrix} y^2 = R^2 - x^2, \\ y^2 = ax - x^3. \end{matrix}$$

The following obvious principle extends this proposition to an extensive and interesting class of problems.

The same value of x which renders y a maximum or minimum in such an equation as (159) *will render it a maximum or minimum whether* (159) *be the equation of a curve or have any other signification.*

Hence, the principle of this proposition serves to determine the maximum or minimum value of any expression involving a variable quantity. Illustrative of this take the following problem:

Ex. 2.—Divide a given straight line into two parts whose rectangle will be the greatest possible.

FIG. 26.

Let AB $= a$, the given line, C the point of division.

$$AC = x, \quad \therefore \ CB = a - x.$$

By putting y for the rectangle of the parts of the line, we have,

$$(e)\qquad y = ax - x^2,$$

the same as equation (a) above. Differentiate (e) and equating its differential coefficient to zero, as in (c), we have,

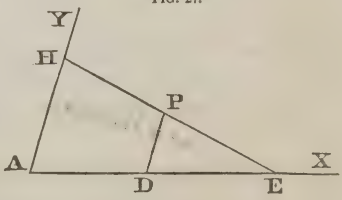
$$(f)\qquad x = \frac{a}{2},$$

which shows that the line is bisected.

As another example, take the following problem:

Ex. 3.—Though a given point in a given angle draw a line, so as to form with the sides of the angle the least possible triangle.

FIG. 27.



Let A be the given angle. P the given point.

$$\left.\begin{matrix} AD = a \\ PD = b \end{matrix}\right\} \text{ given co-ordinates of P.}$$

Put DE $= x$, and by similar triangles, we have,

$$(g)\qquad x : b :: x + a : AH = \frac{b}{x}(x + a),$$

Putting y for the area HAE, we have, by the rule for the area of a triangle,

$$(h) \qquad y = \frac{(a+x)^2}{x} \cdot \frac{b \sin. A,}{2}.$$

Differentiate (h) and putting the differential coefficient equal to zero we have,

$$(k) \qquad \frac{2x(a+x) - (a+x)^2}{x^2} \cdot \frac{b \sin. A}{2} = o.$$

Solve (k) for x, and we have, $x = a$, which shows that the area of the triangle AHE, is the least possible when AD = DE.

Comparing (h) and (k) we observe that the constant factor,

$$\frac{b \sin. A}{2},$$

remains in differentiating, and divides away when the differential cofficient is equated to zero. This factor might, therefore, be omitted before differentiation, without affecting the result.

Again, since any power or root of a quantity is obviously a maximum when the quantity itself is a maximum, the power or root may be omitted before differentiating. Thus, $(ax - x^2)^n$, is obviously a maximum when $ax - x^2$ is a maximum. These considerations might be expressed in the following rules:

A.

A constant factor of a quantity to be made a maximum or minimum, may be omitted before differentiating.

B.

If the quantity to be made a maximum or minimum, be *all* under a given power or root, the power or root may be omitted before differentiating.

C.

Since the Logarithm of a quantity increases or decreases as the quantity itself increases or decreases, an expression to be made a maximum or minimum, may be put into Logarithms before differentiating.

D.

When the differential coefficient is zero, the differential is also

zero. Hence when a quantity is a maximum, or minimum, its differential is zero.

E.

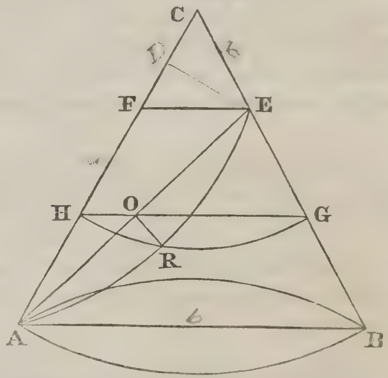
In solving problems in Maxima, or Minima, the first process is to obtain an algebraic expression for the quantity to be made a maximum or minimum. This algebraic expression will be a function of some variable. Differentiate this expression and put its differential equal to zero. This furnishes an equation from which the value of the variable of which the expression is a function, may be obtained.

The determination of the expression for the quantity to be made a maximum, or minimum, is frequently the most difficult part of the process. This is, however, the work of Algebra and Geometry. The Differential Calculus lends its aid only after this expression is obtained.

Illustrative of the foregoing rules and remarks, take the following example.

Ex. 4.—Cut the greatest ellipse from a given cone.

FIG. 28.



Let CAB be the given cone. Put

c = cosine of C,

and

a=AC, b=AB, x=CE.

AE is the trace on the plane of the paper, of the plane that cuts the cone. AE is the major diameter of the ellipse, and AER the half of the ellipse.

Cut the cone by a plane passing through O, the centre of the ellipse, and parallel to the base. The section is a circle, the half of which is HRG. This circle passes through the extremities of the minor diameter of the ellipse. Hence OR perpendicular to the plane of the paper, at O, passes through the point R, the intersection of the ellipse and circle, and OR is the semi-minor diameter.

By Trigonometry, we have, from the triangle CAE,

$$(l) \qquad \mathrm{AE} = (a^2 + x^2 - 2\,cax)^{\frac{1}{2}},$$

and from the triangle CFE, we have,

$$(m) \qquad \mathrm{EF} = (\overline{\mathrm{CF}}^2 + \overline{\mathrm{CE}}^2 - 2\mathrm{CF.CE}\cos.\,\mathrm{C})^{\frac{1}{2}} = x\,(2 - 2c)^{\frac{1}{2}}.$$

Since AO is the half of AE, we have, from the similar triangles AEF and AOH, HO = EF $\div$ 2, or

$$(n) \qquad \mathrm{HO} = \frac{x}{2}\,(2 - 2c)^{\frac{1}{2}}.$$

Also, by similar triangles, EOG and EAB, we have,

$$(o) \qquad \mathrm{OG} = \frac{b}{2}.$$

Since RO is an ordinate of the semi-circle HRG, we have, by Geometry, and the values (n), and (o),

$$(p) \qquad \mathrm{RO} = x^{\frac{1}{2}}.\ \frac{b^{\frac{1}{2}}}{2}.\,(2 - 2c)^{\frac{1}{4}}.$$

Equations (l) and (p) make known the diameters of the ellipse. The rectangle of the diameters multiplied by a constant $\pi \div 4$. is the area of the ellipse. Hence we have,

$$(q) \qquad \therefore \text{area of ellipse} = \frac{\pi}{4}.\,b^{\frac{1}{2}}\,(2 - 2c)^{\frac{1}{4}}\,(a^2x + x^3 - 2cax^2)^{\frac{1}{2}}.$$

So far the process is one of Algebra and Geometry. To apply the Calculus, omit the constant factor in (q), and the radical, and we have,

$$(r) \qquad a^2x + x^3 - 2cax^2,$$

to make a maximum.

Put the differential of (r) equal to zero, and solve the equation for x. We then have,

$$(s) \qquad x = \frac{2}{3}\,ac \pm \frac{a}{3}\,(4c^2 - 3)^{\frac{1}{2}}.$$

Equation (s) determines the position of the vertex E of the ellipse required. If

$$4c^2 = 3, \text{ then} \qquad c = \left(\frac{3}{4}\right)^{\frac{1}{2}}, \text{ and the angle} \qquad \mathrm{C} = 30^\circ.$$

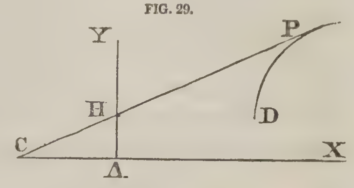
If in (s) we have $4c^2$ less than 3, or the vertical angle greater than 30°, x is imaginary, or such a cone does not admit of a maximum ellipse.

We will revert to this example in a subsequent Proposition.

We will add a few more examples of a general nature, under this Proposition.

Ex. 5,—Determine the point on a curve through which if a tangent line be drawn, the triangle formed by the tangent and co-ordinate axes will be a minimum.

FIG. 29.



Let PD be the curve, and PC the tangent line required. Then AHC is the triangle whose area is a minimum, and P is the point to be determined.

Let (x, y) be the co-ordinates of P, and represent the equation of PD by

$$(2m) \qquad y = \phi x.$$

Then as in Proposition XX, (110), (111), we have for AC and AH,

$$(2n) \qquad AC = x - \frac{\phi x}{\psi x} \text{ and } AH = \phi x - x\psi x.$$

Then putting A for twice the area of AHC, we have, by the rule for the area of a triangle.

$$(2p) \qquad A = \left(x - \frac{\phi x}{\psi x}\right)(\phi x - x\psi x),$$

Here the area is a function of x, the abscissa of P. Differentiate $(2p)$ and this differential put equal to zero (by Rule D, above) furnishes one equation which, with $(2m)$, the equation of the curve, makes known the co-ordinates x, y, of P.

In applying this process to any given curve it would generally be more simple to get, first the equation of the tangent line for the particular curve, and then get the intercepts AH, AC, and form the area AHC, which may always be expressed in terms of one variable by means of the equation of the curve, whether that equation be explicit or implicit.

Ex.—Draw a tangent line, cutting off with the axes, the least triangle, when the curve is the circle with the origin at the centre.

Here $(2m)$ becomes

$$(2q) \qquad x^2 + y^2 - R^2 = o,$$

and for the tangent line to the circle, we have,

$$(2r) \qquad x' x + y' y - R^2 = o,$$

From $(2r)$ find the length of the intercepts AC and AH, and we have for double the area of the triangle,

$$A = \frac{R^4}{xy},$$

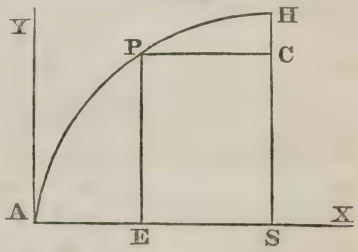
from which one of the variables x or y may be eliminated by equation $(2q)$, and the differential being put equal to zero, furnishes one equation, which with $(2q)$, makes known x and y, the co-ordinates of the point required. In this example x and y are found to be equal.

Ex.—Draw such a tangent line to the curves,

$$a^2 y^2 + b^2 x^2 - a^2 b^2 = o, \qquad a^2 y^2 - b^2 x^2 + a^2 b^2 = o.$$

Ex. 6.—In a given curve inscribe the greatest rectangle.

FIG. 30.



Let (x, y) be the co-ordinates of P, and,

$$(t) \qquad y = \phi x,$$

be the given curve AH.

Let AS$=a$, a known length,

$$\therefore ES = a - x.$$

Then the area of the rectangle PS, is obviously,

$$(u) \qquad PS = (a - x)\, y,$$

or putting for y its value in (t), we have,

$$(2u) \qquad PS = (a - x)\, \phi x.$$

Differentiate $(2u)$, and its differential put equal to zero, furnishes an equation which, with (t), makes known the point P, and determines the rectangle.

Determine such a rectangle in the curves.

$$y = ax, \qquad y = (mx)^{\frac{1}{2}}, \qquad y = (R^2 - x^2)^{\frac{1}{2}}.$$

Ex. 7.—Determine the greatest cylinder that can be inscribed in a given surface of revolution.

If the curve AH, and the inscribed rectangle PS, revolve round AX, AH will generate a surface of revolution, and PS will generate the inscribed cylinder. The radius of the base of the cylinder is obviously PE. Let (x, y) be the co-ordinates of P. Then πy^2 expresses the base of the cylinder, and the solidity of the cylinder is,

$$(v) \qquad \text{cylinder} = \pi y^2 (a - x) = \pi (\varphi x)^2 (a - x).$$

Equate the differential of (v) to zero. This equation, with (t), makes known the point P, which determines the cylinder. Determine such a cylinder when the surface of revolution is generated by any of the curves,

$$x^2 + y^2 - R^2 = o, \ y = ax, \ y^2 = px, \ a^2 y^2 + b^2 x^2 - a^2 b^2 = o.$$

Before leaving this proposition, we will explain a process which is often found convenient.

Resume the simple example:—*Divide a given straight line so that the rectangle of the parts will be a maximum.* Putting a for the length of the line, x and z for its parts, and y for the rectangle, we have,

$$(w) \qquad x + z = a,$$

$$(2a) \qquad y = xz.$$

If we eliminate z from $(2a)$, by means of (w), we have equation (e), as before. But, instead of first eliminating z from $(2a)$, differentiate $(2a,)$ and since $(2a)$ is a maximum, we have,

$$(2b) \qquad xdz + zdx = o.$$

Differentiate also (w), and we have,

$$(2c) \qquad \therefore dx + dz = o.$$

By means of (w) and $(2c)$ eliminate z and dz from $(2b)$. The resulting equation will contain dx in every term, which being divided away and the equation solved for x, we have,

$$x = \frac{a}{2}.$$

This process may always be followed when the expression to be made a maximum or minimum contains two variables, as in $(2a)$, and these variables are related in another equation, as in (w).

The nature of the problem, in most cases, indicates whether it is a maximum or minimum, which is given by the foregoing processes. An analytical test will be given in a subsequent proposition, by which the maxima can be distinguished from the minima values.

When the ordinate admits of neither maxima nor minima values, the algebraic expression itself generally makes known the fact, or it can be readily inferred from the equation obtained by equating the differential coefficient to zero. Thus, if the example were,

Find the maximum ordinate in a parabola,

the equation of the parabola, $y^2 = 4mx$, shows that y is greatest when x is greatest, and least when x is least; consequently there is no ordinate which is greater or less than the ordinate immediately on each side of it. If we differentiate this equation, its differential coefficient put equal to zero, is

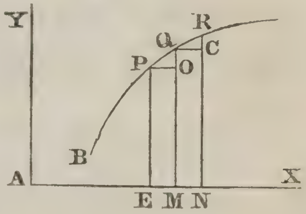
$$\frac{2m}{y} = o, \text{ or } \qquad \frac{m}{x} = o.$$

The first requires y to be infinite, the second requires x to be infinite. Consequently, according to the definition, there is no maximum or minimum ordinate in the parabola.

PROPOSITION XXIX.

Given the equation of a curve, determine its second differential.

FIG. 31.



Let BR be the given curve, and

(161) $\qquad y = ax^3 + b,$

its equation, in which let x,y be the co-ordinates of P.

We have seen, (equation (1) to (9), Proposition I.,) that the differential of (161) is

(*a*) $$dy = 3ax^2. dx,$$

where dy may be represented by QO, and dx by EM. Make

$EM = MN = dx$. Put $y' = QM$, then AM, the abscissa of Q, is $x + dx$.

We observe in (a), that the differential of the ordinate of BR, for the point P, equals the constant $3a$ into the square of the abscissa of P, into dx.

In like manner, the differential of BR, for the point Q, is

$$(b) \qquad dy' = 3a\,(x + dx)^2\,dx,$$

where dy' may be represented by RC, and dx by MN.

Expand (b), and subtract (a) from it, and we have,

$$(c) \qquad dy' - dy = 6ax.dx^2 + 3a.dx^3.$$

The first side of (c) is the difference of the increments of two ordinates, indefinitely near to each other, and is what is understood by a second differential.

Let us represent $dy' - dy$ by d^2y, which is read, *The second differential of* y, and then dividing (c) by dx^2, we have

$$(d) \qquad \frac{d^2y}{dx^2} = 6ax + 3a.dx.$$

Now in the limit, or when dx is indefinitely small, the second side of (d) reduces to the term $6ax$, and (d) becomes

$$(e) \qquad \frac{d^2y}{dx^2} = 6ax.$$

If this be cleared of fractions, we have,

$$(f) \qquad d^2y = 6ax.dx^2.$$

Equation (f) is the second differential of (161). It is obvious that (f) is the differential of (a), on the hypothesis that dx is a constant. But in these processes, x is the independent variable. *Hence to obtain the second differential, differentiate the first differential, making the differential of the independent variable constant.*

Ex. $\quad y = ax^n + C, \therefore \quad dy = nax^{n-1}\,dx$, and

$$d^2y = n\,(n - 1)\,ax^{n-2}\,dx^2.$$

In like manner, we would get the third differential by differentiating the second differential, making dx constant. The forms d^3y, d^4y, d^ny, are used to denote the third differential, the fourth differential, &c. of y.

Equation (e) is called the second differential coefficient.

Differentiate (e), and divide by dx, and we have,

$$(g) \qquad \frac{d^3y}{dx^3} = 6a,$$

the third differential coefficient.

Differentiating (g), and dividing by dx, we have,

$$(h) \qquad \frac{d^4y}{dx^4} = o.$$

Hence the fourth differential coefficient of (161) is zero.

Cor.—It is obvious that between (161), (a), and (f), we could eliminate the two constants a, and b, that enter into (161). The result of this elimination would be, a differential equation of the curve, involving no constant. It is also obvious that if (161) contained three constants, we would have to employ the first, second, and third differentials to eliminate them, &c.

Eliminate a between equations (161), and (a), and we have,

$$(k) \qquad y = \frac{dy}{dx} \cdot \frac{x}{3} + b,$$

which is also the first differential of (161).

Eliminate a and b between (161), (a) and (f), and we have,

$$(l) \qquad \frac{d^2y}{dx^2} \cdot \frac{x}{2} = \frac{dy}{dx},$$

which is, like (f), the second differential of (161).

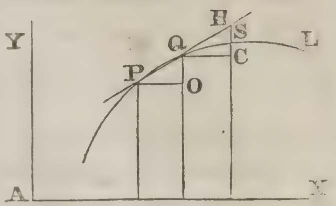
PROPOSITION XXX.

If a curve be concave towards the axis of abscissas, its second differential is negative. If it be convex towards the axis of abscissas, its second differential is positive.

First.—When the curve PL is concave towards the axis of abscissas, let P be any point on it.

F

FIG. 32.



Let PO = QC = dx.

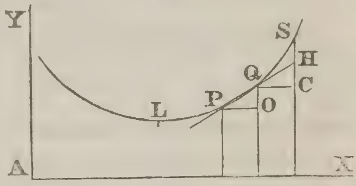
Join the points P and Q, and produce PQ to meet SC in H. It it obvious that PQ must rise above the curve at Q. H is therefore above S, and SC is less than HC. But since PO = QC, we have HC = QO. Also, $dy' - dy$, or its equal, d^2y, as used in the previous Proposition, is HS, that is,

(161a) $$d^2y = SC - QO = SC - HC = -HS.$$

Hence when the curve is concave, the second differential is negative.

Second.—When the curve PL is convex towards the axis of abscissas, let P be any point on it.

FIG. 33.



Make PO = QC = dx.

Join P and Q, and produce PQ to meet SC in H.

At Q, PQ descends *below* the curve. Reasoning as before, we have

(161b) $$d^2y = SC - QO = + SH.$$

Hence if the curve be convex towards the axis of abscissas, the second differential is positive.

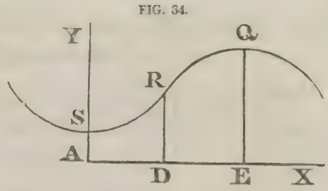
Ex.—Is the curve, $y = ax + x^2$, concave or convex towards the axis of x?

Ans.—Convex.

Cor. 1st.—As the differential coefficient has the same sign as the differential, the second differential coefficient is negative when the curve is concave, and positive when the curve is convex towards the axis of x.

Cor. 2d.—This Proposition enables us to determine when a curve given by its equation is concave or convex towards the axis of x, and also gives us the means of determining the point where it changes from convex to concave.

FIG. 34.



Thus let the equation of the curve SQ be,

(c) $y = ax^2 - x^3 + b.$

The second differential coefficient of this is

(d) $$\frac{d^2y}{dx^2} = 2a - 6x.$$

Equation (*d*) shows the second differential coefficient to be positive, when $2a$ is greater than $6x$, and negative when $2a$ is less than $6x$. Consequently the curve SQ is convex for all values of x less than $a \div 3$, and concave for all values of x greater than $a \div 3$. Consequently at the point R, whose abscissa, $AD = a \div 3$, the curve changes from convex to concave.

The point R, where this change occurs, is called a *singular point*, and is evidently determined by putting the second differential coefficient of the curve equal to zero, for when x equals $a \div 3$, the second side of equation (*d*) is zero.

And in general, if the equation of SQ be

(161*h*) $$y = \varphi x,$$

this equation, in conjunction with the equation,

(161*k*) $$\frac{d^2y}{dx^2} = o,$$

must determine the co-ordinates x and y of the singular point, if the curve have such a point. If the value of the second differential coefficient, from (161*h*), does not contain either of the co-ordinates, it is obvious that (161*k*) does not exist. Thus, to illustrate by a particular case, if the equation of the curve be,

(e) $$y = ax + x^2,$$

then the second differential coefficient is

(f) $$\frac{d^2y}{dx^2} = 2.$$

If this second differential coefficient be put equal to zero, we have $2 = o$, which is absurd. Hence the curve (*e*) has no singular point.

Ex.—Determine the singular points of the curves,

$$y = ax^2 - x^4, \qquad y = ax^3 - x^4, \qquad y = ax^2 - x^5.$$

PROPOSITION XXXI.

When a curve has a maximum ordinate, the second differential coefficient is negative, and when it has a minimum ordinate, the second differential coefficient is positive.

It is obvious that when a curve is concave towards the axis of abscissas, it may have a maximum ordinate at some point L, (see Fig. 32, last Proposition,) but cannot have a minimum ordinate. In Proposition XXX., it was shown that when the curve is concave towards the axis of abscissas, the second differential coefficient, at any point of the curve, is negative. Consequently at the point whose ordinate is a maximum, the second differential coefficient is negative.

It is also obvious, that when a curve is convex towards the axis of abscissas, it may have a minimum ordinate at some point L; (see fig. 33, last Prop.) but cannot have a maximum ordinate. Consequently, by what was shown in Proposition XXX., at the point whose ordinate is a minimum, the second differential, and second differential coefficient are negative.

These principles afford an obvious test to determine the maxima and minima values of algebraic expressions, and enable us to distinguish the one from the other.

(*a*) Ex. 1.—Does the curve $y = ax - x^2$, admit a maximum or minimum ordinate?

Here the first differential coefficient is,

$$(b) \qquad \frac{dy}{dx} = a - 2x,$$

which, put equal to zero, gives,

$$x = \frac{a}{2}.$$

The second differential coefficient is,

$$(c) \qquad \frac{d^2y}{dx^2} = -2,$$

which being negative for all values of x, shows that the curve admits of a maximum and not of a minimum ordinate.

If the second differential coefficient (c) contains x, the values of

x derived from the first differential coefficient put equal to zero must be substituted into (c). If the first differential coefficient put equal to zero, furnishes an equation above the first degree, the values of the variable x derived from it must be severally substituted into (c). Some of these values may render (c) positive, and others of them negative. In such a case the curve admits of both maxima and minima ordinates.

(d) Ex. 2. $$y = ax - x^3 + b.$$

The first differential coefficient of (d) put equal to zero, gives the values,

(e) $$x = + \left(\frac{a}{3}\right)^{\frac{1}{2}}, \text{ and}$$

(f) $$x = - \left(\frac{a}{3}\right)^{\frac{1}{2}}.$$

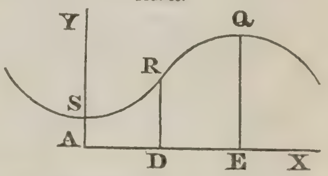
Also, the second differential coefficient of (d) is,

(g) $$\frac{d^2y}{dx^2} = -6x.$$

If in (g) we substitute the value of x in (e), the second differential coefficient is negative. If in (g) we substitute the value of x in (f), the second differential coefficient is positive.

Hence, (d) has both a maximum and a minimum ordinate. The maximum ordinate is given by the value of x in (e), and the minimum ordinate by the value of x in (f).

FIG. 35.



The form of a curve that has both a maximum and minimum ordinate, may be seen in fig. 35, in which AS would be a minimum, and EQ a maximum ordinate.

Such a curve must be convex where the ordinate is a minimum, and concave where it is a maximum; consequently, every curve that has both a maximum and minimum ordinate, must have a *singular point* between the maximum and minimum ordinate points.

From the same figure we may infer generally, that when there are two values of x given by the equation made by putting the

differential coefficient equal to zero, one of them gives a minimum, and the other a maximum ordinate. For, it is obvious in the figure, that there cannot be two maximum ordinates without an intervening minimum one, nor the reverse. Since any algebraic or transcendental expression may be regarded as some power of the ordinate of a curve, the principle of this proposition serves to distinguish the maxima and minima values of all algebraic or transcendental expressions.

As an illustration, recur to Example 4, Proposition XXVIII, which proposed,—*To cut the greatest ellipse from a given cone.* Putting y equal to the expression (r), in that proposition we have for the second differential coefficient,

$$(h) \qquad \frac{d^2y}{dx^2} = 6x - 4ca.$$

And the two values of (s) in that proposition may be written separately.

$$(k) \qquad x = \frac{2}{3}ac + \frac{a}{3}(4c^2 - 3)^{\frac{1}{2}}.$$

$$(l) \qquad x = \frac{2}{3}ac - \frac{a}{3}(4c^2 - 3)^{\frac{1}{2}}$$

If for x in (h) we substitute its value in (k), the second differential coefficient is positive, and if for x in (h) we substitute its value in (l), the second differential coefficient is negative. Hence, the value of x in (k) gives a minimum ellipse, and its value in (l), a maximum ellipse.

If the equation of the curve be,

$$(m) \qquad y = \varphi x,$$

it might occur that the values of x deduced from

$$(n) \qquad \frac{dy}{dx} = o \text{ and } \frac{d^2y}{dx^2} = o,$$

would be the same.

When this is the case the tangent line to the curve at the singular point is parallel to the axis of x. For the first of (n) exists at the point whose tangent is parallel to the axis of x, and the second of (n) exists at the singular point of the curve (m). These points coincide when (m) and the first of (n) give the same values of x and

y that are given by (m) and the second of (n). If the two equations (n) exist, but not together, the singular point does not coincide with the point whose ordinate is a maximum.

Ex.—Determine the maximum ordinate and singular point of the curve whose equation is,

$$(p) \qquad y = ax^2 - x^3.$$

The first and second differential coefficients of this put equal to zero we have the equations,

$$(q) \qquad \frac{dy}{dx} = 2ax - 3x^2 = o, \quad \therefore x = o, \text{ or } x = \frac{2a}{3},$$

$$(r) \qquad \frac{d^2y}{dx^2} = 2a - 6x = o \quad \therefore x = \frac{a}{3}.$$

Equation (q) gives two values of x, and equation (r) a value of x different from both values in (q). Hence, the singular point in (p) does not coincide with the point whose ordinate is a maximum or a minimum.

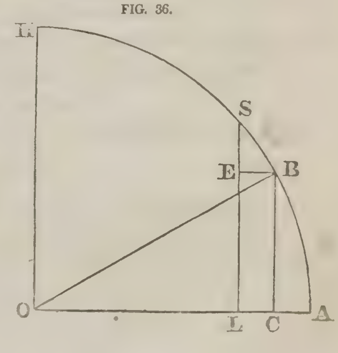
The value $x = o$ gives the minimum ordinate AS and the value $x = a \div 3$, gives the singular point R; and the value $x = 2a \div 3$ gives the maximum ordinate QE.

PROPOSITION XXXII.

Determine the differential of a circular function.

We defined a circular function to be one involving the variable in an arc or trigonometrical line, as $\sin.x$, $\cos.x$, &c. We will first determine the differential of the sine of an arc.

FIG. 36.



Let the arc AB $= x$, BC $= y$, and OA $= 1$ the radius,

$$(162) \qquad \therefore y = \sin.x.$$

Suppose the arc AB receive the increment BS $= h$. Put $y' =$ SL, and we have,

$$(163) \qquad y' = \sin.(x + h).$$

Expand (163) by the usual trigonometrical development, subtract (162) from it, divide by h and we have,

(164) $$\frac{y' - y}{h} = \frac{\sin.h.}{h}.\cos.x + \sin.x.\frac{\cos.h}{h} - \frac{\sin.x}{h}.$$

When an arc is indefinitely small it may be regarded as equal to or coinciding with its sine, and its cosine may be regarded as equal to or coinciding with the radius, therefore, when the increment h is indefinitely small, we have,

$$\sin.h = h, \quad \cos.h = 1 = \text{radius}, \quad y' - y = dy, h = dx.$$

These conditions reduce (164) to

(165) $$\frac{dy}{dx} = \cos.x,$$

or multiplying by dx,

(166) $$dy = \cos.x.dx.$$

Equation (166) is the differential of (162). Comparing (162) and (166), we have,

RULE X.

The differential of the sine of an arc is the cosine of the arc into the differential of the arc.

Ex.— $y = \sin.mx \quad \therefore\ dy = \cos.mx.mdx,$

$y = \sin.^n x \quad \therefore\ dy = n\ \sin.^{n-1}x.\cos.xdx,$

$y = \sin.(m\,x + nx^2).$

Next let us determine the differential of the cosine of an arc.

By trigonometry, we have,

(a) $$\sin.^2 x + \cos.^2 x = 1.$$

Differentiate this by previous rules, and we have,

(b) $$2\sin.x\cos.xdx + 2\cos.xd.\cos.x = o,$$

where $d.\cos.x$ designates that $\cos.x$ is to be differentiated.

By transposition and division, we have from (b).

(c) $$d.\cos.x = -\sin.xdx.$$

Equation (c) furnishes

RULE XI.

The differential of the cosine of an arc is minus the sine of the arc into the differential of the arc.

Ex.— $y = \cos.mx \quad \therefore\ dy = -\sin.mx.mdx,$

$y = \cos.^n x \quad \therefore\ dy = -n\ \cos.^{n-1}x.\sin.xdx,$

$y = \cos.(mx + c).$

To differentiate the versed sine of an arc, we observe by figure 36, that

(d) $$\text{versin.}x = 1 - \cos.x.$$

Differentiate this, and we have,

(e) $$d.\ \text{versin.}x = \sin.x dx,$$

which furnishes the proper rule.

To determine the differential of the tangent of an arc, we know that,

(f) $$\tan.x = \frac{\sin.x}{\cos.x}.$$

Differentiate (f) as a fraction and reduce by (a), and we have,

(g) $$d.\ \tan.x = \frac{dx}{\cos.^2 x} = \text{secan.}^2 x.\ dx,$$

which furnishes the rule.

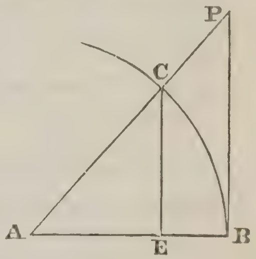
Exs. $y = \tan.(x^2 + a) \quad \therefore \quad dy = \text{secan.}^2(x^2 + a).\ 2xdx.$

$$y = \sin.x \tan.x.$$

Such an example is differentiated as the product of two factors, $\sin.x$ being one factor, and $\tan.x$ the other.

We have hitherto supposed the trigonometrical lines to be functions of the arc. Let us reverse this process, and consider the arc as a function of the trigonometrical line.

FIG. 37.



Let AB = 1, the radius.

Let the arc BC = y, and the sine line CE = x, the cosine line AE = z, and the tangent line BP = u. Then

(h) $x = \sin.y,\ z = \cos.y,\ u = \tan.y.$

In order to express the arc y, in terms of the trigonometrical lines, the equations (h) may be written,

(k) $y = \sin.^{-1}x, \qquad y = \cos.^{-1}z$, and $\qquad y = \tan.^{-1}u,$

where $\sin.^{-1}x$ is a mere arbitrary notation employed to designate the arc of which x is the sine line. In like manner, $\cos.^{-1}z$ designates the arc of which z is the cosine line, &c. To differentiate (k), take the direct forms, (h). From the first of (h), we have,

(l) $$dy = \frac{dx}{\cos.y}. \quad \text{But} \quad \cos.y = \text{AE} = (1 - x^2)^{\frac{1}{2}}.$$

Hence,

(m) $$dy = \frac{dx}{(1 - x^2)^{\frac{1}{2}}}.$$

Equation (m) furnishes the rule for differentiating the first of (k), viz :

RULE XII.

The differential of an arc, in terms of the sine line, is the differential of the sine line, divided by the square root of radius squared, minus the sine line squared.

Proceed in the same way with the second of (h), and we have,

(n) $$dy = -\frac{dz}{(1 - z^2)^{\frac{1}{2}}},$$

which furnishes the rule for differentiating the second of (k), viz :

RULE XIII.

The differential of an arc in terms of the cosine line, is minus the differential of the cosine line divided by the square root of radius squared, minus the cosine line squared.

Proceed in the same way with the third of (h), and we get,

(o) $$dy = \frac{du}{1 + u^2},$$

which furnishes the rule for differentiating the third of (k), viz:

RULE XIV.

The differential of an arc in terms of the tangent line, is the differential of the tangent line divided by the radius squared, plus the tangent line squared.

If $v =$ the versed sine EB, then we have,

(p) $$v = \text{versin}.y, \text{ and } \quad y = \text{versin}.^{-1}v.$$

Differentiate the first of (p), and we have,

(q) $$dy = \frac{dv}{\sin.y} = \frac{dv}{(2v - v^2)^{\frac{1}{2}}},$$

which furnishes the rule for differentiating an arc in terms of the versed sine.

Exs.—Differentiate the equations,

$y = \text{versin.}^{-1}.\,av,$ $\qquad y = \text{versin.}^{-1}v^2,$

$y = \tan.^{-1}.(x + a),$ $\qquad y = \tan.^{-1}(x + a)^2,$

$y = \cos.^{-1}(x + a),$ $\qquad y = \sin.^{-1}(x + a).$

Illustrative of the preceding rules, we subjoin the following examples, which are solved by the principles of Proposition XXVIII., and in which circular functions are employed.

Ex. A.

Divide a right angle so that the rectangle of the sines of the parts will be a maximum.

Let x equal one part of the angle, then $90° - x$, is the other, and we have,

(r) $\qquad$ Rectangle $= \sin.x.\ \sin.(90° - x) = \sin.x \cos.x.$

Differentiate (r), as the product of two factors, by Rules X. and XI., and putting the differential zero, we have,

(s) $\qquad \cos.^2x.dx - \sin.^2x.dx = o.$

From this we have,

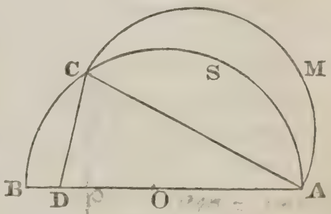
$$\cos.x = \sin.x,$$

which shows that the angle is bisected.

Ex. B.

In a given semicircle, inscribe the greatest isosceles triangle, one of whose sides coincides with the diameter, and whose vertex is on the perimeter of the circle.

FIG. 38.



Let the radius of the circle be, OA = 1, and let CAD be the required triangle.

Let the arc ASC $= 2x$. Let AC and AD be the sides of the isosceles triangle. Put

(t) $\quad y = \text{AC} = \text{AD}, \quad \therefore \quad y = 2\sin.x,$ and $\text{CAB} = 90° - x.$

(u) $\qquad \therefore \quad \sin.\text{CAB} = \cos.x.$

Put z for the area of the triangle, CAD, and we have, by the rule for the area of a triangle,

$$(v) \qquad z = y^2 . \frac{\cos.x}{2}.$$

Substitute into (v), the value of y in (t), and we have,

$$(w) \qquad z = 2 \sin.^2 x. \cos.x = 2(\cos.x - \cos.^3 x.$$

Differentiate (w) by Rule XI., and the differential being equated to zero, and solved, we have,

$$(x) \qquad \cos.x = \left(\frac{1}{3}\right)^{\frac{1}{2}}, \text{ or } \qquad \sin.x = o,$$

The first gives a maximum, the second a minimum triangle, as may be determined by Proposition XXXI.

Ex. C.

In a given semicircle, draw a chord AC, (Fig. 38), such that if a semicircle CMA be described on the chord, the lune CMAS will be the greatest possible.

Taking the same data as in Ex. B, and putting z for the area of the lune, we readily obtain from Plane Geometry, and Trigonometry, the expression for the area, viz :

$$(2a) \qquad z = \tfrac{1}{2} \pi \sin.^2 x + \sin.x \cos.x - x.$$

The differential of $(2a)$, being put equal to zero, we have,

$$(2b) \qquad \pi \sin.x \cos.x - 2 \sin.^2 x = o.$$

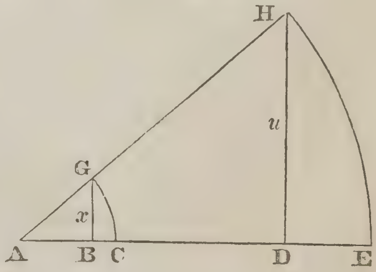
This is satisfied by making,

$$(2c) \qquad \sin.x = o, \text{ or } \qquad \pi \cos.x - 2 \sin.x = o.$$

The first gives a minimum, the second a maximum lune, as may be determined by Proposition XXXI.

In the deduction of the previous rules, we supposed the radius of the circle to be a unit.

FIG. 39.



Suppose, however, we have to employ an arc HE, whose radius is AH $=$ R.

Take AG, a radius equal to unity, and describe an arc GC. Draw the sine lines HD and GB of these arcs.

Put GB$=x$, and HD$=u$.

By similar triangles, AGB and AHD, we have,

(2*d*) $$1 : x :: R : u,$$

from which we get,

(2*e*) $$\therefore x = \frac{u}{R}.$$

Also, since similar arcs are as their radii, we have, from the similar arcs GC and HE,

(2*f*) $$1 : GC :: R : HE.$$

By the notation for an arc in terms of its sine line, we have, from the fig.

(2*g*) $$GC = \sin.^{-1} x = \sin.^{-1} \frac{u}{R}.$$

Substitute this value of GC into (2*f*), and we have,

(167) $$HE = R \sin.^{-1} \frac{u}{R}.$$

Equation (167) expresses that the arc, whose radius is R, is R times the similar arc whose radius is unity.

If z be the tangent of the arc HE, we have, in like manner,

(168) $$HE = R \tan.^{-1} \frac{z}{R}.$$

Proceeding in the same manner, we would obtain similar values when the arc is a function of any other trigonometrical line.

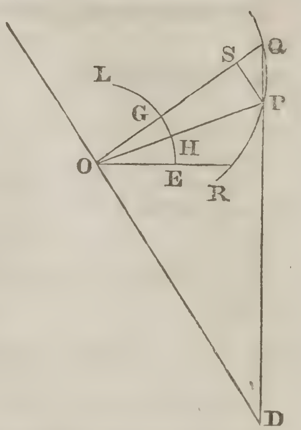
PROPOSITION XXXIII.

Determine the polar subtangent of a curve.

Definition.—If a line be drawn through the pole perpendicular to the radius vector to the point of tangency, the part of this line between the pole and tangent is the *polar subtangent.*

The method of changing the equation of a curve from rectangular to polar co-ordinates, is explained in Analytical Geometry. Many curves, as the spinals, &c., are more readily discussed by referring them to polar co-ordinates.

FIG. 40.



Let O be the pole, OE the angular axis, OP the radius vector, P the point of tangency, PR the curve referred to polar co-ordinates, PD the tangent, and OD perpendicular to OP, then is OD the polar subtangent.

To find the length of OD, in terms of constants, and the co-ordinates of P, take the radius OE $= 1$, and describe the arc EL.

Let the point Q be indefinitely near to P, draw OQ, and with centre O, and radius OP, describe the arc PS, then we have, OS $=$ OP.

In the limit, or when PQ and PS are indefinitely small, they may be taken as straight lines, and the angle S may be taken a right angle. Since EH is the measure of the angle POE, put

(A). $\quad \mathrm{EH} = \omega, \quad \mathrm{OP} = \mathrm{R}. \quad$ Then $\mathrm{HG} = d\omega, \quad \mathrm{SQ} = d\mathrm{R}.$

Since similar arcs are as their radii, the arcs HG and PS give,

(169) $$1 : d\omega :: \mathrm{R} : \mathrm{PS} = \mathrm{R}d\omega.$$

The tangent PD may be regarded as coinciding with the curve along the indefinitely small arc PQ, and the triangles QSP and POD are similar, from which we have,

(170) $$\therefore \mathrm{QS} : \mathrm{SP} :: \mathrm{PO} : \mathrm{OD}.$$

By means of (169), and (A), this becomes,

(171) $$d\mathrm{R} : \mathrm{R}d\omega :: \mathrm{R} : \mathrm{OD}.$$

Therefore, the subtangent is,

(172) $$\mathrm{OD} = \mathrm{R}^2 . \frac{d\omega}{d\mathrm{R}}.$$

If now the polar equation of the curve RP be represented by the form,

(173) $$\mathrm{R} = \varphi\omega,$$

we deduce from (173) the value of

$$\frac{d\omega}{d\mathrm{R}},$$

and substitute it into (172).

The result is the length of the polar subtangent, involving the co-ordinates of the point of tangency. (See Prop. F, App.)

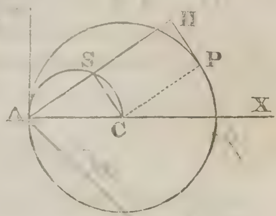
Ex.—The polar equation of the circloid (136*f*) is,

$$(a) \qquad R = r \cos.\omega + r,$$

find the polar subtangent. The value of the differential coefficient deduced from (a) and put into (172), we have, for the subtangent

$$(c) \qquad OD = -\frac{R^2}{r \sin.\omega}.$$

FIG. 41.



Since $r \sin.\omega = CS$, the subtangent of the circloid is a fourth proportional to the three lines CS, AH, and AH. Hence, the subtangent may be constructed, and the tangent to circloid drawn.

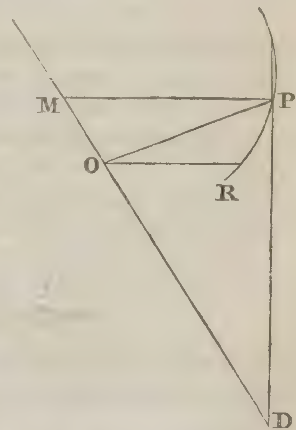
Ex. 2.—Find the polar subtangent in the curves.

$$\left.\begin{array}{l} R = \dfrac{m}{1 + e.\cos.\omega}, \quad R = c\omega^2, \\ R = \log.\omega, \quad R^2 = c^2 \cos.^2\omega + b^2, \end{array}\right\} \text{H.}$$

PROPOSITION XXXIV.

Determine the polar subnormal of a curve.

FIG. 42.



If PM be drawn perpendicular to the tangent PD, and the subtangent OD be produced, OM is the polar subnormal.

Because in the right angled triangle MPD, PO is drawn perpendicular to the hypotenuse MD, we have PO, a mean proportional between MO and OD, that is,

$$(174) \qquad \overline{PO}^2 = OD \,.\, OM,$$

substitute into this the value of OD, in (172), and we get, since $PO = R$,

$$(175) \qquad OM = \frac{dR}{d\omega},$$

which is the length of the subnormal required.

We have, then, (173) from which to get the differential coefficient used in (175).

Ex. 1.—Find the polar subnormal of the circloid.

The equation of this curve at (136*f*) is,

$$(a) \qquad R = r \cos.\omega + r.$$

The value of $dR \div d\omega$, deduced from (a) and put into (175), we have,

$$(b) \qquad OM = -r \sin.\omega.$$

Hence, the subnormal equals CS, figure 41.

Ex. 2.—Determine the polar subnormal in the curves H.

PROPOSITION XXXV.

Find the length of the polar normal of a curve.

The line PM, figure 42, is the polar normal of the curve PR.

Putting N for the length of this normal, we have, from the right angled triangle POM.

$$N^2 = \overline{PO}^2 + \overline{OM}^2.$$

Put in this for PO, its equal R, for OM, its value in (175), and we have,

$$(175a) \qquad N^2 = R^2 + \frac{dR^2}{d\omega^2},$$

which makes known the length of the normal.

Ex.—Find the length of the normal in the circloid.

Put into (175*a*) for $dR \div d\omega$, its value deduced from the equation of the circloid, and we have,

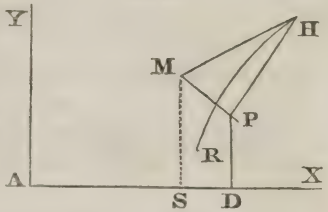
$$(175b) \qquad N^2 = 2r\,(r + r \cos.\omega) = 2\,rR,$$

which shows the normal to be a mean proportional between the diameter of the circle and the radius vector.

PROPOSITION XXXVI.

Determine the locus of the intersections of the tangent and polar subtangent in a plane curve.

FIG. 43.



Let P be the pole, HR the curve, MH the tangent, and MP perpendicular to PH, MP is the polar subtangent, and M is one point of the locus required.

Let (c,d) be the co-ordinates of P.

Let (x,y) be the co-ordinates of H, one point of tangency.

Let (m,n) be the co-ordinates of M.

Get the equation of HP, passing through the points H and P, and the equation of MP passing through P, and perpendicular to HP, is, for the point M,

(177) $$n - d = -\frac{c - x}{d - y}(m - c).$$

The equation of the tangent line MH is, as in (93), for the point M,

(178) $$n - y = p\,(m - x).$$

The equation of the curve RH may be represented by

(179) $$\phi(x,y) = o.$$

At the point H, x and y are common to these three equations. Eliminate these co-ordinates between (177), (178), and (179). The resulting equation will contain no other variables than m and n, and may be represented by

(180) $$\varphi(m,n) = o,$$

which is the equation of the locus required.

If either or both of the co-ordinates, (c,d), of P be zero, (177) must be modified accordingly.

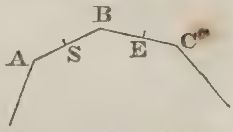
Ex.—Determine this locus when the curve is the ellipse, hyperbola, or parabola, and the focus the pole.

Ans.—A straight line.

CONSECUTIVE LINES AND POINTS.

An idea of consecutive points of a curve may be formed by supposing the curve to be a polygon of an indefinite number of sides, AB, BC, &c.

FIG. 44.



The middle points S and E, of two adjacent sides of this polygon, would be consecutive points on the curve, and are to be regarded as indefinitely near to each other.

If through the two consecutive points S and E, normals be drawn, these normals would be consecutive lines; we may therefore say in general, that consecutive lines, (straight or curved), are lines indefinitely near to each other, fulfilling given conditions. Thus in the case of the consecutive lines through S and E, the conditions to be fulfilled were, that they be normals to the curve.

The following Proposition, and Examples under it, will elucidate the doctrine of Consecutive Lines.

PROPOSITION XXXVII.

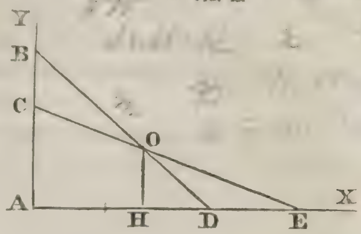
A line, (straight or curved), is drawn subject to a given condition, determine the locus of the intersection of this line with its consecutive line.

This general proposition may be more readily understood by examining a few particular cases under it. Take first, the following.

Ex. A.

A straight line is drawn, cutting off, with the sides of a given angle, a triangle of given area, determine the locus of the intersection of this line with its consecutive line.

FIG. 45.



Let AX and AY be the sides of the given angle, and the axes of reference. Let CE be drawn, cutting off the triangle CAE of given area. This is the condition to which the line CE is subject. Let the equation of CE be

(181) $$y = ax + \beta.$$

As this equation contains two parameters, a and β, let us seek, by the condition to which the line is subject, to eliminate one of them. For this purpose put s = area of CAE, and we have, from (181), for

$$x = o, \quad y = \beta = \text{AC, and for} \quad y = o, \quad x = -\frac{\beta}{a} = \text{AE}.$$

Then for the area we have,

(182) $$s = -\frac{\beta^2}{a} \cdot \frac{\sin.\text{A}}{2}.$$

Substitute the value of a from (182) into (181), and we have, after putting m for $\sin.\text{A} \div 2s$,

(183) $$y = -m\beta^2 x + \beta.$$

This is the equation of CE, containing one parameter, β, and conditioned to cut off a triangle, CAE, of a given area. As the parameter β in (183), is arbitrary, it is obvious that the line CE, depending for its position on β, may take an indefinite number of positions fulfilling the condition of cutting off the given area. Suppose BD be a line consecutive to EC, i. e., cutting off a triangle, BAD = CAE, and since BD and CE are indefinitely near to each other, if we put BC = $d\beta$, and increase the β of (183) by $d\beta$, we have,

(184) $$y = -mx\,(\beta + d\beta)^2 + \beta + d\beta,$$

which is the equation of BD, the line consecutive to CE. The point O, where these two lines intersect, is the point whose locus we are to find.

It is obvious that the co-ordinates x and y, are common to (183), and (184), at the point of intersection O. Hence, if we subtract (183) from (184), and divide by $d\beta$, we have,

(185) $$o = -2m\,\beta x + 1 - mxd\beta.$$

If we suppose $d\beta$ indefinitely small, or zero, the term containing it in (185) may be omitted, and we have,

(186) $$o = -2m\,\beta x + 1.$$

In (186) the co-ordinate x must still appertain to the point of intersection O. But when $d\beta$ is indefinitely small, the lines (183) and (184) are what we have called consecutive lines, and the process by which we passed from (183) to (186), is obviously the process for differentiating (183) for x and y constant and β variable; which is also made evident by the fact that if we differentiate (183) for β variable, and divide by $d\beta$, we have (186). Hence, if the position of a line depends upon a parameter, if we differentiate the equation of the line making that parameter the only variable, the co-ordinates x,y of the differential equation will appertain to the point of intersection of the line with its consecutive line.

Since at the point of intersection O the co-ordinates x,y are common to (183) and (186), if we eliminate β between these two equations, we have,

(187) $$xy = \frac{1}{4m}.$$

Equation (187) is the locus of the point O of intersection, that is, if the two consecutive lines CE and BD be imagined to move round within the given angle, subject to the condition that each always cuts off a given triangle, the locus of their intersection is equation (187). The curve (187) is tangent to the line CE in every position, for it may be conceived to be generated by a point O moving along CE as CE changes its position. (See Proposition E, Appendix. Cor.)

From this process, we may infer generally, that if

(188) $$\phi(x,y,\beta) = o,$$

be the equation of a line, straight or curved, whose position depends upon the parameter β; if we differentiate (188) for β variable, and x,y constant, the co-ordinates x,y of the differential will appertain to the point of intersection of (188) with the consecutive line. Represent the differential of (188) for β variable, and divided by $d\beta$, by

$$(189) \qquad \frac{d.\phi(x,y,\beta)}{d\beta} = o.$$

Eliminate β between (188) and (189), and the resulting equation in x and y, which may be represented by

$$(190) \qquad \mathrm{F}\,(x,y) = o,$$

will be the locus of the intersection of (188) with its consecutive line. The line (190) is tangent to the line (188), for it may be conceived to be generated by the point of intersection (as O in fig. 45,) moving along (188).

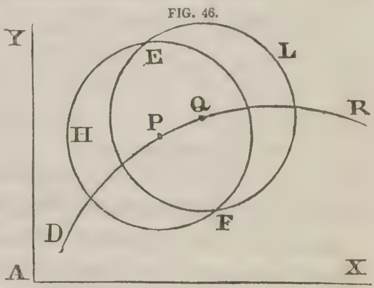
(See Proposition E, Appendix.)

We will illustrate this by several other examples.

Ex. B.

The centre of a circle moves along a given curve, find the locus of the intersection of the circle with its consecutive circle.

FIG. 46.



Let DR be the curve on which the centre of the circle moves, and

$$(191) \qquad \beta = \phi a,$$

the equation of DR.

Let P be the centre of the circle PH in one position.

The equation of PH is, (a and β being co-ordinates of P),

$$(192) \qquad (x - a)^2 + (y - \beta)^2 = \mathrm{R}^2,$$

or substituting into (192) from (191),

$$(193) \qquad (x - a)^2 + (y - \phi a)^2 = \mathrm{R}^2.$$

Here a is the parameter on which the position of the circle depends, and by the same reasoning employed in Example A, we could show, that if we differentiate (193) for x and y constant, and the parameter a variable, the co-ordinates x and y in the differential, which may be written,

$$(194) \qquad (x - a)\,da + (y - \phi a)\,d.\phi a = o,$$

will be the co-ordinates of the point E, where the circle PH intersects the consecutive circle QL.

Eliminate a between (194) and (193), and the resulting equation, which will be of the form (190), will be the locus of the point of intersection E. This locus will be tangent to the circle (193), in every position.

To particularise the present example, suppose the line RD be a straight line, then (191) becomes

$$(a) \qquad \beta = ma.$$

Substitute this into (192), and differentiate for a variable, and we have,

$$(b) \qquad -2(x-a)\,da - 2m(y-ma)\,da = o.$$

Eliminate a between (b) and (192), and we have,

$$(c) \qquad y = mx \pm R(m^2+1)^{\frac{1}{2}}$$

which shows that the locus is two parallel lines, one of them being the locus of E, and the other of F.

If the centre had moved along the axis of abscissas, (192) would have been,

$$(d) \qquad (x-a)^2 + y^2 = R^2.$$

Differentiate this for a variable, and eliminate a between (d), and this differential, and we have,

$$(e) \qquad y = \pm R,$$

which represents two parallel lines, whose distance apart is 2R.

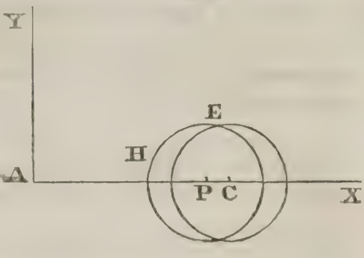
Instead of first eliminating β from (192), by means of (191), and then differentiating the result (193), we might first have differentiated (192) for a and β variable, and then have eliminated β and $d\beta$ by means of (191) and its differential. This process, which we will have occasion hereafter to adopt, is the same as that shown in Proposition XXVIII., (w), ($2c$), &c.

Ex. C.

The centre of a circle moves along a given straight line, its radius varies as the distance of the centre from a given point in that line; find the curve to which it is always tangent.

Take the given line as the axis of abscissas.

FIG. 47.



Let the origin A be the given point, and a the distance from the origin to the centre P of the circle in one position.

Since the radius varies as the distance a, let $R \propto a \therefore \frac{R}{a} = m$

(d) $$R = ma,$$

be the radius where m is any numerical quantity.

The equation of the circle PH is therefore,

(e) $$(a - x)^2 + y^2 = m^2a^2.$$

Differentiate this for a variable, gives

(f) $$a - x = m^2a.$$

Eliminate a between (f) and (e), and we have,

(g) $$y = \pm \frac{mx}{(m^2 - 1)^{\frac{1}{2}}}$$

an equation which shows that the circle is always touched by two straight lines passing through the origin.

Ex. D.

The centre of a circle moves along a given straight line, its radius varies as the square root of the distance of the centre from a fixed point on that line. Find the curve to which the circle is always tangent.

Take the data as in Ex. C, and since the radius varies as the square root of the distance of the centre from the origin, we may express this law of variation by

(e) $$R = \sqrt{ma}.$$

and the equation of the circle is,

(f) $$(a - x)^2 + y^2 = ma.$$

Eliminate a between (f) and its differential for a variable, and we have a parabola.

We would proceed in the same manner if the radius varied as any given function of the distance of the centre from the origin. For putting,

(g) $$R = \phi a,$$

the equation of the moving circle is,

$$(h) \qquad (x - a)^2 + y^2 = (\phi a)^2.$$

Eliminate a between (h) and its differential, for a variable the result is the locus sought.

Ex. E.

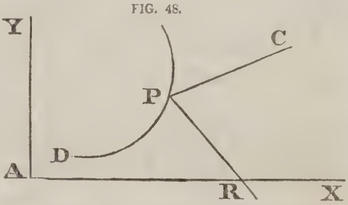
The vertex of a parabola moves along a given curve, its axis remains parallel to itself. Find the curve touching the parabola in every position.

This is readily done by imitating the steps in Ex. B.

Ex. F.

A given angle moves along a given curve, one of its sides passes through a given point, find the curve to which the other side is always tangent.

FIG. 48.



Let PD be the given curve, CPR the given angle. Let the given point be R on the axis of abscissas. Let (a,β) be the coordinates of P, put AR $= c$ and $v =$ tan.CPR.

The equation of PR passing through the two points P and R, is

$$(k) \qquad y = \frac{\beta}{a - c}(x - c).$$

The equation of PC passing through P, and making the given angle with PR, is by Analytical Geometry.

$$(l) \qquad y - \beta = \frac{v(a - c) + \beta}{a - c - v\beta}\left(x - a.\right\}$$

Let now,

$$(m) \qquad \beta = \phi a,$$

be the equation of PD. Eliminate β from (l) by means of (m), and then eliminate a between (l) and its differential for a variable, the result is the locus required.

If the angle CPR, is a right angle, then v is infinite and (l) becomes.

$$(n) \qquad y - \beta = \frac{c - a}{\beta}(x - a),$$

which is the equation of PC in this case.

To derive (n) from (l), divide the numerator and denominator of the fraction in (l) by v, and then putting v infinite (l) becomes (n).

Ex.—If the curve PD becomes a straight line coinciding with the axis of y, then $a = o$ and (n) becomes

$$(o) \qquad y - \beta = \frac{c}{\beta}\, x.$$

Eliminate β between (o) and its differential for β variable, and we have,

$$(p) \qquad y^2 = 4cx,$$

a parabola, for the locus sought.

When PD becomes a straight line coinciding with the axis of y, equation (l) becomes

$$(q) \qquad y - \beta = \frac{vc - \beta}{c + v\beta}.\, x,$$

which is the equation of PC. Find the curve to which PC is tangent in this case. *Ans.* A Parabola.

What is the locus when PD is a circle, with centre at the origin, the angle CPR being a right angle.

If r be the radius of the circle, the equation of the locus is,

$$(r) \qquad r^2\, y^2 + x^2\,(r^2 - c^2) - r^2\,(r^2 - c^2) = o,$$

which is a circle when c is zero, an ellipse when c is less than r, a hyperbola when c is greater than r, and a point when c is equal to r.

This Proposition is the converse of Proposition XXIV.

Ex. G.

Between the sides of a right angle a straight line of given length is drawn, determine the curve to which this line is always tangent.

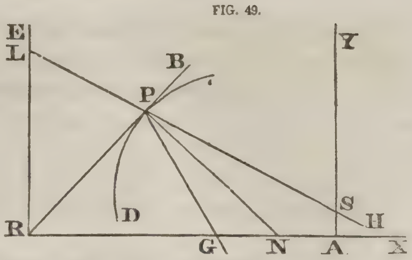
Put m for the length of the given line and we find for the curve sought, the equation,

$$y^{\frac{2}{3}} + x^{\frac{2}{3}} = m^{\frac{2}{3}},$$

Ex. H.

An indefinite number of parallel lines are drawn to meet a given curve, where each parallel meets the curve, a line is drawn, making with the parallel, an angle which is bisected by the normal at that point; find the locus of the intersection of this line with its consecutive line.

FIG. 49.



Let PD be the curve, SL one of the system of parallels meeting the curve in P.

Suppose PN the normal at P, and PG the line drawn making the angle GPN = SPN. Draw BR tangent at P, and RE parallel to the axis of y. Let v equal the tangent of the given angle ELP, which each parallel makes with the axis of y. Let

$$(s) \qquad \beta = \phi a,$$

be the curve PD.

The equation of the normal PN is,

$$(t) \qquad y - \beta = -\frac{1}{p}(x - a).$$

The equation of PG is of the form,

$$(u) \qquad y - \beta = b\,(x - a),$$

and it remains to express b in terms of constants, and of the co-ordinates of P. For this purpose, we have,

$$(2a) \qquad \text{tan.(ELP — LRP)} = \text{tan.LPR} = \text{tan.BPS} = \text{cot.GPN}.$$

$$(2b) \qquad \therefore \quad \text{tan.GPN} = \frac{1}{\text{tan.(ELP — LRP)}} = \frac{p + v}{pv - 1}.$$

Again, since b is the tangent of the angle PGN, we have,

$$(2c) \qquad b = \text{tan.PGN} = \text{tan.(PNX — GPN)}.$$

Expand the last side of (2c), and by means of (2b), and recollecting that

$$\text{tan.PNX} = -1 \div p,$$

we have, for b, the value,

$$(2d) \qquad b = \frac{2pv + p^2 - 1}{2p - v\,(p^2 - 1)}.$$

Substitute this value of b into (u), and we have for the equation of PG,

$$(2e) \qquad y - \beta = \frac{2pv + p^2 - 1}{2p - v\,(p^2 - 1)}\,(x - a).$$

By means of (s), β may be eliminated from $(2e)$, and the remaining parameter a being eliminated between $(2e)$ and its differential for a variable, the resulting equation is the locus sought.

Cor. 1st.—If the parallels SL be parallel to the axis of abscissas, AX, then v is infinite, and $(2e)$ becomes

$$(2f) \qquad y-\beta=\frac{2p}{1-p^2}(x-a),$$

which is, in this case, the equation of PG.

Cor. 2d.—If the parallels SL be parallel to the axis of ordinates AY, then v is zero, and equation $(2e)$ becomes

$$(2g) \qquad y-\beta=\frac{p^2-1}{2p}(x-a),$$

which is, in this case, the equation of PG.

Proceed with $(2g)$, or $(2f)$, as directed for $(2e)$, and we have the locus sought.

Example H comprehends the Theory of Caustic Curves, the rays of light being regarded as parallel, and the law of reflexion of light being that the incident and reflected rays make equal angles with the normal, at the point of incidence.

For a detailed investigation of these curves, see the Analysis of the Marquis de L'Hopital, on the subject, in which they are examined with much neatness and elegance, and from more elementary considerations than those here employed.

As an example, suppose the given curve be the parabola,

$$y^2=4ma$$

and the lines be parallel to the axis of abscissas, find the proposed locus.

Using $(2f)$, we have, for the curve required,

$$y^2+(x-m)^2=o,$$

which designates a point, viz : the focus of the parabola.

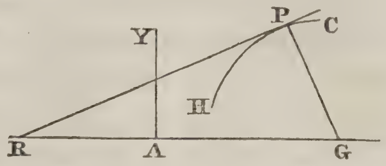
If instead of being parallel, the lines SP were drawn from a point, as at S, and made the same angle with the normal PN, which the normal made with PG, we could, by a similar process, obtain the equation of PG, involving as parameters the co-ordinates of P, and one of these parameters being eliminated from the equation of PG,

by means of the equation of the curve, we could, as before, find the curve to which PG is always tangent. This curve would be the caustic when the rays of light emerge from a point.

Ex. K.

A tangent is drawn to a given curve, at the point of tangency a line is drawn making a given angle with the tangent, determine the locus of the intersection of this line with its consecutive line.

FIG. 50.



Let PH be the given curve, and its equation

(2h) $\quad y = \phi x.$

The equation of the tangent line PR, is

(2k) $$y' - y = p\,(x' - x),$$

where x',y' are the co-ordinates of any point on PR. Let PG be the line drawn, making with PR, at the point P, a given angle RPG, whose tangent we will represent by v. The equation of PG, passing through the point P, is

(2m) $$y' - y = \frac{p + v}{1 - pv}\,(x' - x),$$

where x and y are the co-ordinates of the point of tangency, and x',y', the co-ordinates of any point on PG. By means of the curve (2h), eliminate one of the parameters x,y, from (2m), and we have,

(2n) $$y' - \phi x = \frac{p + v}{1 - pv}\,(x' - x).$$

Putting in this the value of p, in terms of x, deduced from (2h), we eliminate x between (2n), and the differential of (2n) for x variable. The resulting equation, which may be represented by

(2p) $$\phi(x',y') = o,$$

is the locus required.

Suppose the given curve be the circle,

$$x^2 + y^2 = R^2.$$

Then proceeding as above, we have for the locus,

$$x'^2 + y'^2 = R^2\,(\cos.RPG)^2$$

Ex. L.

A line of given length moves with its extremities along two given curves, determine the locus of its intersection with its consecutive line.

Ex. M.

A parabola passes through a given point, and has its vertex on a given curve, and its axis perpendicular to a given line, find the locus of its intersection with its consecutive parabola.

Ex. N.

A rectangle is constructed on the ordinate and abscissa of a given curve, a diagonal is drawn from the foot of the ordinate, find the locus of its intersection with its consecutive diagonal.

PROPOSITION XXXVIII.

Determine the radius of curvature at any point in a given curve.

Definitions.

1.—A circle which coincides with a curve at two consecutive points, is called, *an Osculating Circle.*

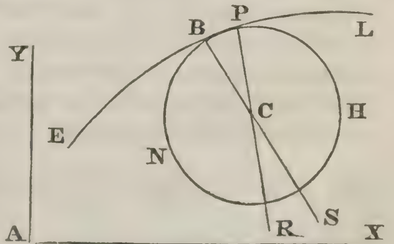
2.—The radius of the osculating circle is called, *the Radius of Curvature.*

3.—The point where the radius of curvature meets the curve is called, *the Point of Osculation.*

4. The centre of the osculating circle is called *the centre of curvature.*

Let EL be the curve, B and P consecutive points on it, and let the circle HBN coincide with EL at B and P, then C is the centre of curvature, CB the radius of curvature, and HBN is the osculating circle.

FIG. 51.



In order to determine the radius of curvature at any point B of EL, proceed as follows. Let (a,β) be the co-ordinates of any point on EL, and let the Equation of EL be represented by

(195) $$\beta = \phi a.$$

Through the consecutive points B and P, draw BS and PR, normals to EL. Since (a,β) are the co-ordinates of any point B on EL, the equation of the normal BS is, (see 67).

(196) $$\beta - y + \frac{da}{d\beta}(a - x) = o.$$

If we differentiate (196), for the parameters a and β variable, the x and y of the differential, as was shown in the previous proposition, will be the co-ordinates of C the intersection of BS with its consecutive normal PR. The differential of (196) for a and β variable, a being the independent variable is,

(197) $$d\beta - da.\frac{d^2\beta}{d\beta^2}(a - x) + \frac{da^2}{d\beta} = o.$$

The consecutive normals BS and PR intersect at the centre of the osculating circle. For since the curve EL and the circle BH coincide at P and B, the normals to the curve at B and P are likewise normals to the circle, and normals to the circle all pass through its centre.

If x and y be the co-ordinates of the centre C, we have for the length of the radius of curvature BC, calling it R,

(198) $$R = \left((\beta - y)^2 + (a - x)^2\right)^{\frac{1}{2}}.$$

At the centre C, the co-ordinates x and y are common to (196), (197) and (198). Hence, find the values of x and y in (196) and (197), and substitute them into (198), and we have the length of the radius of curvature in terms of the co-ordinates of the point of osculation B, and their differentials.

The length of the radius of curvature found by eliminating x and y between these equations is,

(199) $$R = \pm\left(1 + \frac{d\beta^2}{da^2}\right)^{\frac{1}{2}} \div \frac{d^2\beta}{da^2} \text{ or } R = \pm\frac{(1 + p^2)^{\frac{3}{2}}}{r},$$

the last form being made from the first by putting,

$$\frac{d\beta}{da} = p \text{ and } \frac{d^2\beta}{da^2} = r.$$

This radius being a function of the co-ordinates of the point of osculation will depend on the nature of the curve EL, and vary when the point of osculation is varied. To find it for any particular curve (195), deduce the first and second differential coefficients from (195), and substitute their values in (199).

Ex. 1. Let the equation of the curve EL be,

(*a*) $$\beta = ma^2,$$

what is the radius of curvature?

(*b*) Here $\frac{d\beta}{da} = 2ma = p$ and $\frac{d^2\beta}{da^2} = 2m = r$,

substitute these into (199), and we have, for the radius of curvature,

(*c*) $$\text{R} = \frac{(1 + 4m^2a^2)^{\frac{3}{2}}}{2m}.$$

Here the radius of curvature is a function of a, the abscissa of the point of osculation, and evidently increases as that abscissa increases. If the abscissa a be zero (*c*) gives for the radius of curvature at the origin.

(*d*) $$\text{R} = \frac{1}{2m}.$$

Ex. 2. Find the radius of curvature in the following curves,

$$\beta = ma^3, \qquad \beta^2 = 4ma, \qquad \beta^2 = ma + na^2,$$

in which a is the abscissa and β the ordinate.

The radius of curvature (199) is positive or negative. These signs only indicate the direction in which the radius is measured in respect of the curve. We have seen, Proposition XXX, that if the curve be concave towards the axis of abscissas, the second differential coefficient is negative. In order, therefore, that the radius of curvature may be positive in such curve, (199) must be taken with the negative sign. For a like reason, if the curve be convex towards the axis of abscissas, (199) must be taken with the positive sign.

Since the osculating circle coincides with the given curve, at the point of osculation, the curvature of the curve, at that point, is the

same as the curvature of the osculating circle, and is therefore known, if we know the radius of curvature.

Before leaving this Proposition, it may be proper to observe, that the radius of curvature (199) is deduced on the supposition that a is the independent variable, and consequently, (see Proposition XXIX.) da constant. Equation (196) was differentiated on this hypothesis. If, on the contrary, β be the independent variable, then $d\beta$ is constant, and da variable, in differentiating (196), (see Proposition XXIX.) In order to obtain a form that will embrace both of these hypotheses, differentiate (196), as if neither a nor β were the independent variable, that is, consider da and $d\beta$ both variable in (196). Then eliminate x and y from (198), by means of (196), and its differential for both da and $d\beta$ variables, and we have, for the radius of curvature,

$$(199a) \qquad R = \pm \frac{(da^2 + d\beta^2)^{\frac{3}{2}}}{d^2\beta da - d^2a.d\beta}.$$

If da be constant, or $d^2a = o$, (199a) becomes (199).

If $d\beta$ be constant, or $d^2\beta = o$, (199a) shows the proper value of the radius of curvature on that hypothesis. If a and β be both functions of some other variable quantity, taking that quantity as the independent variable, (199a) shows the length of the radius of curvature. We will have occasion to employ this last principle in determining the radius of curvature of a polar curve.

PROPOSITION XXXIX.

Given a curve, determine its evolute.

Definitions.

1.—The locus of the centre of curvature is called, ***the Evolute of a Curve.*** Thus the locus of C, fig. 51, is the evolute of the curve EL.

2.—The given curve, when reference is had to the evolute, is called, ***the Involute.*** Thus EL, fig. 51, is the involute of which the locus of the centre C is the evolute.

To determine the evolute, let EL be the given curve, and (195)

its equation. We have seen, in Proposition XXXVIII., that the centre C is the intersection of the normal (196), with its consecutive normal. Hence, eliminate the parameters a and β between the three equations, (195), (196), and (197), and the resulting equation, which may be represented by

$$\phi(x,y) = o, \tag{200}$$

is the evolute required.

To perform this elimination most readily, first solve (196) and (197) for x and y, and we have

$$x = a - \frac{p(1+p^2)}{r}. \tag{201}$$

$$y = \beta + \frac{1+p^2}{r}. \tag{202}$$

Substitute into (201) and (202) the values of the differential coefficients deduced from (195), and then from these two resulting equations and (195), eliminate a and β.

This Proposition is evidently merely a corrollary to Ex. K, Proposition XXXVII. For when in Ex. K, the angle RPG becomes a right angle, PG becomes a normal, the locus becomes the evolute, and ($2n$) becomes,

$$y' - \phi x = -\frac{1}{p}(x' - x), \tag{202a}$$

from which, if x be eliminated, as in Proposition XXXVII., we have the evolute.

Ex.—Determine the evolute of the curves,

$\beta^2 = 4ma,$	the paraboia,
$a\beta = m^2,$	the hyperbola,
$m^2\beta^2 + n^2a^2 = m^2.n^2,$	the ellipse.

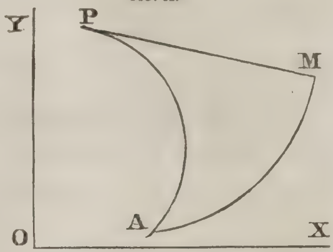
PROPOSITION XL.

Normals to the involute are tangents to tne evolute.

This is a corollary from the foregoing Propositions ; for the evolute being the locus of the intersection of consecutive normals, this locus, by the Theory of Consecutive Lines, in Proposition XXXVII.

(see Proposition E., Appendix), is tangent to the normal, at its intersection with its consecutive normal. Hence the truth of the Proposition.

FIG. 52.

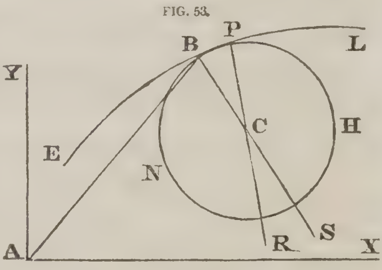


From this it follows, that if a cord be wound round a given curve AP, and then unwound, the unwinding beginning at a point A, and the cord being kept tense, and in the plane of the curve AP, the extremity of the cord will describe the involute AM, of which AP is the evolute. It is for this reason that the locus of the intersection of consecutive normals is called the evolute.

PROPOSITION XLI.

Determine the radius of curvature when the curve is referred to polar co-ordinates.

FIG. 53.



Suppose EL be the curve. Let the origin A, of rectangular axes, be the pole, and let the axis of abscissas AX be the angular axis. If (a,β) be the co-ordinates of any point B on EL, the rectangular equation of EL is, as in (195),

(203) $$\beta = \phi a.$$

Put ρ for the radius vector AB, and ω for the variable angle BAX, then the values of a and β in (203), are

(204) $$a = \rho\cos.\omega, \qquad \beta = \rho\sin.\omega.$$

Here, since a and β are both functions of ρ and ω, we may take ω as the independent variable, and obtain the first and second differentials of a and β in (204). Substitute these differentials into

(199a), and we have the length of the radius of curvature in terms of polar co-ordinates. The result of this substitution is,

(205) $$R = \pm \frac{(\rho^2 + u^2)^{\frac{3}{2}}}{\rho^2 + 2u^2 - \rho u'},$$

where $$u = \frac{d\rho}{d\omega} \text{ and } u' = \frac{d^2\rho}{d\omega^2}.$$

We should have had the same result if instead of taking the axis of abscissas for the angular axis, we had taken any line inclined to that axis at an angle c. This is evident, inasmuch as the variable angle enters (205) only in the form of a differential. It may also be proved by taking instead, of (204), the values

$$a = \rho\cos.(\omega - c), \qquad \beta = \rho\sin.(\omega - c).$$

By substituting the differentials from these equations into (199a), we have, as before, (205).

Ex. 1. Find the radius of curvature in the circloid.

Putting ρ for the radius vector, and m for the radius of the circle the equation of the circloid (136f) is,

(a) $$\rho = m\cos.\omega + m.$$

Differentiating this, we have,

(b) $$\frac{d\rho}{d\omega} = -m\sin.\omega = u \text{ and } \frac{d^2\rho}{d\omega^2} = -m\cos.\omega = u'.$$

Substitute the differential coefficients (b) into (205), and reducing by (a), we have for the radius of curvature R, the value,

(c) $$R = \tfrac{2}{3}(2m^2 + 2m^2\cos.\omega)^{\frac{1}{2}} = \tfrac{2}{3}(2m\rho)^{\frac{1}{2}},$$

Comparing (c) with (175b), we see that the radius of curvature in the circloid, is two-thirds of the polar normal.

Ex. 2. Find the radius of curvature of the curves.

(K) $$\begin{cases} \rho = c\omega^2, & \rho^2 = c^2\cos.^2\omega + b^2. \\ \rho = m\log.\omega, & \rho = \dfrac{m}{1 + e\cos\omega}. \end{cases}$$

PROPOSITION XLII.

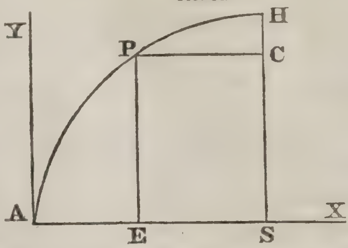
Lemma.

In any equation of the form,

(206) $$y' = \varphi(x \pm h),$$

the differetial coefficient is the same for x constant and h variable, as for h constant and x variable.

FIG. 54.



If we put in the curve AH,

(A) $\quad AE = x, \quad PE = y,$

$ES = h, \quad HS = y',$

equation (206) may be regarded as the equation of the curve AH, and it is evident in the curve that y' varies for h constant and x variable in the same manner and degree that it does for x constant and h variable. This would give us

(206a) $$\frac{dy'}{dx} = \frac{dy'}{dh}.$$

As a particular case, let (206) be,

(a) $$y' = (x + h)^n.$$

Differentiate this for h constant and x variable, and we have,

(b) $$\frac{dy'}{dx} = n\,(x + h)^{n-1}.$$

Differentiate (a) for x constant, and h variable, and we have,

(c) $$\frac{dy'}{dh} = n\,(x + h)^{n-1}.$$

The value of the differential coefficients (b) and (c), are the same.

PROPOSITION XLIII.

Taylor's Theorem.

Determine a general method of expanding into a series, a function of the sum or difference of two independent variables.

Let us represent the equation of the curve AH by

(207) $$y = \varphi x.$$

If x be increased by h, the equation of the curve is,

(208) $$y' = \phi\,(x + h),$$

in which h may be represented by ES, and y' by HS.

The object of the proposition is to expand the second side of (208). For this purpose let $\phi(x+h)$ be assumed equal to a series in the ascending powers of h, viz.

(209) $\phi(x+h) = A + Bh + Ch^2 + Dh^3 + \&c.,$

or, which is the same, put

(209a) $y' = A + Bh + Ch^2 + Dh^3 + \&c.,$

where A, B, C, &c., are indeterminate coefficients containing x but not h.

It remains to determine the coefficients A, B, C, &c., and their values being put into (209), we will have the expansion of $\phi(x+h)$.

To determine these coefficients, we observe, that the first term of (209) being independent of h will remain unchanged, whatever be the value of h, and consequently, will not change though $h = o$.

But when $h = o$, (208), and (209a), become (207), consequently we have from (209), or (209a),

(210) $A = y = \phi x.$

To determine the other coefficients, B, C, &c., differentiate (209a) for h variable, and x, (i. e. A, B, C, &c.) constant, and then differentiate (209) for h constant and x variable, and since by (206a) of the preceding theorem, we have,

$$\frac{dy'}{dh} = \frac{dy'}{dx},$$

we will have, by equating the values of these differential coefficients from (209a),

(211) $B + 2Ch + 3Dh^2 + \&c. = \frac{dA}{dx} + h.\frac{dB}{dx} + h^2.\frac{dC}{dx} + \&c.$

By the principle of the method of Indeterminate Coefficients, the coefficients of the like powers of h, on each side of (211), may be put equal to each other. This furnishes the equations,

(212) $B = \frac{dA}{dx}.$

(213) $2C = \frac{dB}{dx}.$

(214) $3D = \frac{dC}{dx}, \&c.$

By means of (210), equation (212) becomes,

(215) $$B = \frac{dy}{dx}. \text{ Then } dB = \frac{d^2y}{dx},$$

and (213) becomes,

(216) $$C = \frac{d^2y}{2dx^2}. \text{ Then } dC = \frac{d^3y}{2dx^2},$$

and (214) becomes,

(217) $$D = \frac{d^3y}{2.3.dx^3}.$$

Substitute the values of A, B, C, D, &c., from (210), (215), (216), (217), into (209a), and we have for the development required,

(218) $$y' = y + \frac{dy}{dx} h + \frac{d^2y}{2.dx^2} h^2 + \frac{d^3y}{2.3.dx^3} h^3 + \&c.$$

This serves to determine the ordinate y' in terms of any other ordinate y, of the increment h, and of the differential coefficients of y. In this theorem, h is not supposed indefinitely small, as in the procedure in Proposition I., but is of any value whatever.

Transpose y in (218), and we have,

(219) $$y' - y = \frac{dy}{dx} h + \frac{d^2y}{2.dx^2} h^2 + \frac{d^3y}{2.3.dx^3} h^3 + \&c.$$

The second side of (219) shows the increment HC, in terms of h, and of the co-ordinates of P.

Comparing (207), (208), and (218), the latter equation may be written,

(220) $$\varphi(x+h) = \varphi x + \frac{d\varphi x}{dx}.h + \frac{d^2\varphi x}{2.dx^2}.h^2 + \frac{d^3\varphi x}{2.3.dx^3}.h^3 + \&c.$$

which shows, at a single view, the development of $\varphi(x+h)$.

This theorem is of very extensive application, and is employed by most writers on the Calculus, in resolving the Geometrical Propositions that have occupied our attention in the previous part of this work. We will have occasion to employ it hereafter.

As an application of this development, let us develope

(a) $$y' = (x+h)^n.$$

This is a particular case of (208). By making h zero in (a), we have,

(b) $$y = x^n,$$
which is what (207) becomes in this particular example.

Deduce from (b), the successive differential coefficients, and substitute them into (218), and we have,

(c) $$y' = (x + h)^n = x^n + nx^{n-1}h + n.\frac{(n-1)}{2}x^{n-2}h^2 + \&c.$$

In the same manner, develope

$$y' = \log.(x + h), \qquad y' = \log.(x - h).$$
$$y' = \sin.(x + h), \qquad y' = \tan.(x + h).$$

If equation (208) had been,

$$y' = \phi(x - h),$$

the algebraic signs of (218), (220), would have been alternately positive and negative, i. e. the odd powers of h in (218) or (220), are negative.

PROPOSITION XLIV.

Maclaurin's Theorem.

Determine a general method of expanding into a series a function of a single variable.

Suppose we have the equation,

(221) $$y = \phi x,$$
which may or may not contain constants.

Assume the second side of (221) equal to a series in the ascending powers of x, containing the indeterminate co-efficients A, B, C, &c., that is,

(222) $$y = A + Bx + Cx^2 + Dx^3 + \&c.$$

The second side of (222) is the development of ϕx, and it remains to ascertain the values of A, B, C, &c.

To determine these coefficients, we observe, that (222) being, by hypothesis, the true development, whatever be x, it is therefore, true when $x = o$. If we denote by y' the value of (222), when $x = o$, we have,

(a) $$y' = A.$$

Differentiate (222), and we have,

(b) $$\frac{dy}{dx} = B + 2Cx + 3Dx^2 + \&c.$$

This, also, must be true when $x = o$. When $x = o$, (b) becomes

$$(c) \qquad \frac{dy'}{dx'} = \mathrm{B}, \text{ where by } \frac{dy'}{dx'},$$

we denote the value of $\frac{dy}{dx}$ when x is zero.

In the same way differentiate (b), and put $x = o$, and we have,

$$(d) \qquad \frac{d^2y'}{dx'^2} = 2\mathrm{C} \text{ or } \mathrm{C} = \frac{d^2y'}{2dx'^2},$$

Differentiate (b) twice and put $x = o$, and we get,

$$(e) \qquad \mathrm{D} = \frac{d^3y'}{2.3.dx'^2}.$$

Substitute these values, (a), (c), (d), (e) into (222), and we have,

$$(223) \qquad y = y' + \frac{dy'}{dx'}x + \frac{d^2y'}{2dx'^2}x^2 + \frac{d^3y'}{2.3.dx'^2}.x^3 + \&c.$$

In (223) the accented differential coefficients and y' are used to denote the value of the differential coefficients and original function of x when $x = o$.

As an application of this formula, take the following example.

Ex. 1.—Develope cos.x into a series in terms of x. Put

$$(f) \qquad y = \cos.x,$$

and when $x = o$, $\cos.x = \text{radius, or } y' = 1$.

From (f) by differentiating, we have,

$$(g) \qquad \frac{dy}{dx} = -\sin.x, \text{ and when } x = o, \text{ we have, } \frac{dy'}{dx'} = o.$$

Differentiate (g), and we have,

$$(h) \qquad \frac{d^2y}{dx^2} = -\cos.x, \text{ and when } x = o, \text{ we have, } \frac{d^2y'}{dx'^2} = -1,$$

and so on, the odd differential coefficient being zero when $x = o$, and the even one plus and minus unity alternately. These values substituted into (223), we have,

$$(k) \qquad y = \cos.x = 1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \frac{x^6}{2.3.4.5.6} + \&c.$$

Equation (k) shows the length of the cosine of an arc in terms of the radius (unity) and of the arc x.

After the same manner develope,

$$y = \sin.x, \qquad y = \tan.x, \qquad y = \cos.^{-1}x,$$
$$y = \tan.^{-1}x, \qquad y = \sin.^{-1}x.$$

PROPOSITION XLV.

Determine the equation of a plane tangent at a given point, on a given curve surface.

The equation of a curve surface contains three variables, and may be represented generally by the form,

(224) $$\varphi(x,y,z) = o,$$

which denotes an equation containing three variables.

Particular forms of (224) are

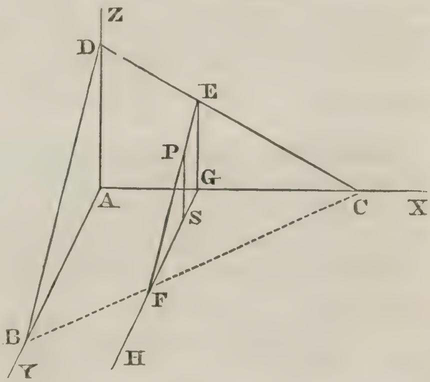
(L) $$\left\{\begin{array}{ll} x^2 + y^2 + z^2 + r^2 = o, & \text{the sphere.} \\ x^2 + y^2 - pz = o, & \text{the paraboloid.} \\ A^2z^2 + B^2x^2 + B^2y^2 - A^2B^2 = o, & \text{the ellipsoid.} \end{array}\right.$$

If the equation of a surface be solved for one variable, as z, it will be of the form,

(225) $$z = \varphi(x,y).$$

We will use either of the forms, (224), (225), according to convenience.

FIG. 53.



Suppose (224) be the given surface, P the given point on it, and (x,y,z) the co-ordinates of P.

If x',y',z', be the variable co-ordinates on a plane, then the equation of any plane DCB, passing through the point P, is of the form

(226) $$z' - z = m(x' - x) + n(y' - y),$$

where

$$m = \tan.\text{DCX}, \qquad n = \tan.\text{DBY},$$

the angles which the traces of the plane, (226), make with the axes.

Suppose plane (226) tangent to surface (224) at P, and that DC and DB are its traces.

Through P pass an auxiliary plane parallel to the plane ZY. This auxiliary plane cuts plane (226), in the right line EF, parallel to the trace DB.

The same auxiliary plane cuts the curve surface (224), making a section whose plane is the auxiliary plane.

This section will have the right line EF for its tangent at P. The traces EG and GH, of the auxiliary plane, are parallel to the axes AZ, AY.

The tangent line EF makes with GH an angle EFH, equal to DBY.

The equation of the section made by plane EGH, is found by making, in (224), x constant, and equal to AG.

The differential coefficient $\frac{dz}{dy}$, deduced from the equation of this section, expresses the tangent of the angle EFH, (Proposition II.)

Therefore we have,

(227) $$\frac{dz}{dy} = n.$$

In the same manner, by passing a plane through P, parallel to the plane ZX, we would have,

(228) $$\frac{dz}{dx} = m.$$

The differential coefficient (227), is supposed to be deduced from the equation of the section made by plane EFH. But as this section is made from surface (224), by making x constant, we get (227) directly, by differentiating (224) for x constant.

In like manner, we get (228) by differentiating (224), supposing y constant.

Substitute (227) and (228) into (226), and we have for the equation of (226),

(229) $$z' - z = \frac{dz}{dx}(x' - x) + \frac{dz}{dy}(y' - y).$$

This plane contains two lines at right angles, tangent to surface

(224). Equation (229) is therefore the tangent plane to surface (224), at the point (x,y,z).

To apply (229) to a given surface, deduce from the proposed surface (224), the differential coefficients, as above directed, and substitute their values into (229).

Ex.—Find the tangent plane to the sphere.

In this case, (224) becomes,

$$(a) \qquad x^2 + y^2 + z^2 - r^2 = o.$$

Differentiating (a) first for x constant and then for y constant, we have,

$$(b) \qquad \frac{dz}{dy} = -\frac{y}{z} \text{ and } \frac{dz}{dx} = -\frac{x}{z},$$

substitute these values into (229), and subtracting (a) from the result, we have, for the plane required,

$$(c) \qquad xx' + yy' + zz' - r^2 = o.$$

Ex. 2. Find the tangent plane to the surfaces,

$$x^2 + y^2 - 4mz = o, \qquad a^2y^2 + a^2x^2 + b^2z^2 - a^2b^2 = o.$$

From the foregoing we observe, that when one of the co-ordinates, as x, in the equation of a surface, is constant, and the other two variable, the variation, differentiation, &c., takes place in the section made by a plane parallel to the co-ordinate plane ZY. Similarly when y is constant, and x and z variable, the variation, differentiation, &c., takes place in a section parallel to the plane ZX. For these reasons, the equation of a surface is an equation containing two independent variables.

In the foregoing and subsequent propositions, we will consider x and y as the independent variables, z being the dependent variable.

The differential coefficient deduced from the equation of a surface, on the supposition that one of the co-ordinates is constant, is called, *The Partial Differential Coefficient*. Thus $\frac{dz}{dx}$ and $\frac{dz}{dy}$ in (b), are partial differential coefficients deduced from the sphere (a).

Let the sections of a curve surface, which are made by planes parallel to the co-ordinate planes, ZX and ZY, be called, *Parallel Sections*, and let the traces on XY, made by the planes of the parallel sections, be called, *Parallel Traces*. The section of the sur-

face (224), made by the plane EGH, is a parallel section, and GH is a parallel trace.

From (227), (228), we observe that the partial differential coefficient of a curve surface expresses the tangent of the angle which the tangent line to a parallel section makes with its parallel trace.

When the partial differential coefficients (227), (228), exist together, it is obvious that the tangent lines are drawn from the intersection of the parallel sections, and lie in the plane tangent to the surface at that point.

PROPOSITION XLVI.

Determine the point on a given curve surface through which, if a tangent plane be drawn, its traces will make given angles with the co-ordinate axes.

Let (224) represent the given surface.

Put m and n for the tangents of the given angles, and we have (224), (227), and (228), three equations to determine x, y, and z, the co-ordinates of the point required.

Ex. 1.—Determine such a point on the surface of a sphere.

Here (224) becomes,

$$(a) \qquad x^2 + y^2 + z^2 - r^2 = o, \quad \therefore \frac{dz}{dy} = -\frac{y}{z} \text{ and } \frac{dz}{dx} = -\frac{x}{z},$$

and (227), (228), become,

$$(b) \qquad -\frac{y}{z} = n, \text{ and } \qquad -\frac{x}{z} = m.$$

Solve the two equations (b), and equation (a), for x, y, and z, and we have,

$$z = \frac{r}{(m^2 + n^2 + 1)^{\frac{1}{2}}}, \; y = \frac{-nr}{(m^2 + n^2 + 1)^{\frac{1}{2}}}, \text{ and } x = \frac{-mr}{(m^2 + n^2 + 1)^{\frac{1}{2}}},$$

for the three co-ordinates required.

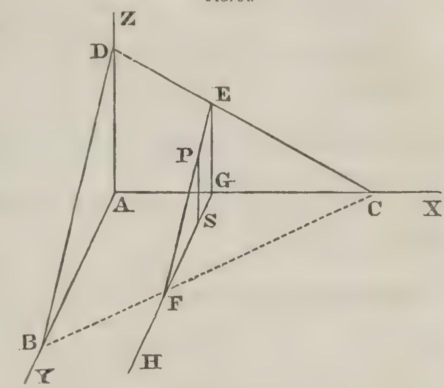
Ex. 2.—Determine such a point on the surfaces,

$$y^2 + x^2 - 4mz = o, \qquad a^2 y^2 + a^2 x^2 + b^2 z^2 - a^2 b^2 = o.$$

PROPOSITION XLVII.

Determine the point on a given curve surface at which the ordinate z is a maximum or minimum.

FIG. 54.



Let the equation of the given surface be,

(230) $\varphi(x,y,z) = o.$

Through the point required, suppose a plane tangent to the given surface be drawn.

The traces of a tangent plane make angles with the axes of X and Y, whose tangents are the partial differential coefficients, (227), (228). But when the ordinate z of the point of tangency is a maximum or minimum, the tangent plane must be parallel to the plane XY. In that case the points B and C are infinitely distant, and the angles DCX and DBY are zero or 180°, and their tangents are zero. Hence, when z is a maximum or minimum, (227) and (228), become

(231) $$\frac{dz}{dy} = o,$$

(232) $$\frac{dz}{dx} = o.$$

Equations (230), (231), and (232), make known the co-ordinates of the point on the surface whose ordinate is a maximum or minimum.

Ex. 1. Determine the maximum or minimum ordinate when the surface is a sphere.

If (a,β,c), be the centre of the sphere, its equation is,

(a) $$(x - a)^2 + (y - \beta)^2 + (z - c)^2 - R^2 = o.$$

By differentiating (a), we have,

(c) $$\therefore \frac{dz}{dy} = -\frac{y-\beta}{z-c}, \text{ and } \frac{dz}{dx} = -\frac{x-a}{z-c}.$$

These differential coefficients being as in (231), (232), put equal to zero, we have,

(d) $$-\frac{y-\beta}{z-c} = o \text{ and } -\frac{x-a}{z-c} = o.$$

We have the three equations (a) and (d) to solve for x, y and z.

These give,

(e) $$x = a, \quad y = \beta, \quad \text{and } z = c \pm R,$$

which are the co-ordinates of the point on (a), whose ordinate z is a maximum or minimum.

Ex. 2. Determine the maximum or minimum ordinate in the surface,

(f) $$z = axy - x^2y - xy^2.$$

Differentiating this first for x, and then for y constant, we have,

(g) $$\frac{dz}{dy} = ax - x^2 - 2xy.$$

(h) $$\frac{dz}{dx} = ay - 2xy - y^2.$$

Put (g) and (h), each equal to zero, as in (231) and (232), and we have,

(k) $$ax - x^2 - 2xy = o, \text{ and}$$

(l) $$ay - 2xy - y^2 = o.$$

Solve (k) and (l) for y and x, and we have,

(m) $$x = \frac{a}{3}, \text{ or } x = o,$$

(n) $$y = \frac{a}{3}, \text{ or } y = o.$$

To distinguish the maxima from the minima ordinates in cases where the nature of the proposition does not immediately make known which is a maximum and which a minimum, we have the following considerations.

When the surface is concave towards the plane XY, it may have a maximum ordinate z drawn from some point on its concave surface, but not a minimum ordinate. If it be convex towards the plane of XY it may have a minimum ordinate drawn from some point on the convex surface to the plane XY, but not a maximum

ordinate. But if a surface at any point be concave towards the plane of XY, parallel sections at that point are concave towards the parallel traces on XY, and conversely if the surface be convex. Hence, by the reasoning in Proposition XXXI, when the surface has a maximum ordinate, we have,

$$\frac{d^2z}{dy^2} = \text{a negative quantity, and}$$

$$\frac{d^2z}{dx^2} = \text{a negative quantity,}$$

and conversely when the surface has a minimum ordinate.

To apply this test to Example 1, differentiate equations (c), and by means of the values in (e), to wit:

$$(o) \qquad x = a, \qquad y = \beta, \text{ and } z = c + \mathrm{R}, \text{ we have,}$$

$$(p) \qquad \frac{d^2z}{dy^2} = -\frac{1}{\mathrm{R}} \text{ and } \frac{d^2z}{dx^2} = -\frac{1}{\mathrm{R}}.$$

By means of the negative value of z in (e) to wit:

$$(q) \qquad x = a, \qquad y = \beta, \text{ and } z = c - \mathrm{R},$$

the second differential coefficients of (c), become

$$(r) \qquad \frac{d^2z}{dy^2} = +\frac{1}{\mathrm{R}} \text{ and } \frac{d^2z}{dx^2} = +\frac{1}{\mathrm{R}}.$$

Hence, the value of the co-ordinates (o) give a maximum, and the value of the co-ordinates (q) a minimum ordinate z.

The same values of x and y that render z a maximum or minimum in the expression,

$$(233) \qquad z = \phi(x,y),$$

will render it a maximum or minimum whether (233) be the equation of a surface, or have any other signification.

This obvious principle extends this Proposition to an interesting class of Problems. As an example, take the following.

Ex. 3.—Divide a given straight line into three parts, so that their product will be the greatest possible.

Let $a =$ the line, and let x,y, and $a - x - y$ be the three parts.

Then putting z for the continued product, we have,

$$(s) \qquad z = axy - x^2y - xy^2,$$

the same as (f), above, and the values of the parts x and y are determined as in the process (f) to (n).

The value of x and y in (m) and (n), show the line to be trisected.

Ex. 4. Divide a given straight line into three parts, such that the product of one part, into the square of the second, into the cube of the third, will be a maximum.

Put z for the product, a for the line, x,y, and $a - x - y$, for the parts, and we have, by the enunciation,

$$z = x^3y^2(a - x - y).$$

Clear of the vinculum, and proceed as in (f) to (n).

Before leaving this Proposition, we will explain a process often found convenient in practice.

Resume the example. *Divide a given straight line into three parts, such that their product will be a maximum.*

Put $a =$ the line, and let x, y, and u be the three parts, and $z =$ the product.

Then by the enunciation, we have,

(2a) $$x + y + u = a, \text{ and}$$

(2b) $$z = xyu.$$

Instead of eliminating u from (2b), by means of (2a), differentiate (2a), and (2b), for x,y and u variable, and we have,

(2c) $$dx + dy + du = o.$$

(2d) $$dz = xydu + xudy + uydx.$$

Eliminate u and du from (2d), by means of (2a) and (2c), and we have,

(2e) $$dz = axdy - x^2dy - 2xydy + aydx - 2xydx - y^2dx.$$

Put the terms on the second side of (2e), that contain dy equal to zero, and the terms that contain dx equal to zero, and we have the equations

(2f) $$ax - x^2 - 2xy = o.$$

(2g) $$ay - 2xy - y^2 = o.$$

These equations are the same as (k) and (l).

This process might be explained generally, but the foregoing will suffice. Illustrative of the process, take the following example.

Ex. 5. Given any number of plane areas situated on given planes, determine the plane on which, if all the given plane areas be orthogonally projected, the sum of the areas of the projections will be the greatest possible.

The planes of the given areas being known in reference to three co-ordinate planes, let the given areas be each projected on each of the co-ordinate planes. Let

M = the sum of their projections on the plane XY.
M' = " " " XZ.
M'' = " " " ZY.

Let x, y, and u be the angles which the plane required makes with the co-ordinate planes XY, XZ and ZY respectively. If z be put for the sum of the projections of the given areas on the plane required, we have, as is shown in books on Analytical Geometry,

$$(2h) \qquad z = M \cos.x + M' \cos.y + M'' \cos.u.$$

By Analytical Geometry, we also have,

$$(2k) \qquad \cos.^2 x + \cos.^2 y + \cos.^2 u = 1.$$

Proceed with $(2h)$, and $(2k)$, as was done with $(2a)$, $(2b)$, and we have for the required angles.

$$\cos.x = \frac{M}{(M^2 + M'^2 + M''^2)^{\frac{1}{2}}}, \quad \cos.y = \frac{M'}{(M^2 + M'^2 + M''^2)^{\frac{1}{2}}},$$

$$\cos.u = \frac{M''}{(M^2 + M'^2 + M''^2)^{\frac{1}{2}}}.$$

The three angles thus determined make known the position of the required plane.

This is *the plane of Greatest Projection*, and is the same that is known in Mechanics by the name of *The Invariable Plane.*

Ex. 6. Given the solidity of a rectangular parallelopipedon, determine the edges when the surface is a minimum.

Ans. The edges are equal.

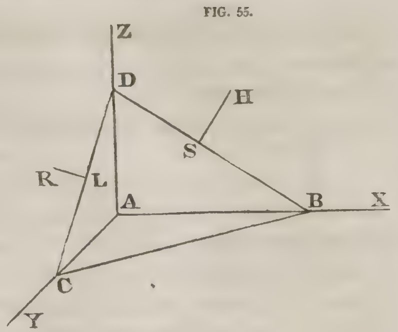
Ex. 7. Determine the three angles of a plane triangle, so that the product or sum of the sines of the angles will be a maximum.

Ans. The angles are equal.

PROPOSITION XLVIII.

Determine the equation of a normal line at a given point on a curve surface.

FIG. 55.



Let the equation of the surface be

(234) $\varphi(x,y,z) = o.$

Suppose P the point on the surface through which the normal is to be drawn. Through P pass a plane tangent to the surface.

Let DC and DB be the traces of such a tangent plane.

If x', y', and z', be the variable co-ordinates of this plane, its equation is, (see 229).

(235) $$z' - z = p\,(x' - x) + q(y' - y),$$

where we have put, for brevity,

(236) $$p = \frac{dz}{dx} \text{ and } q = \frac{dz}{dy},$$

abbreviations frequently used hereafter.

Since the normal to the surface is perpendicular to the tangent plane (235), the projections HS and RL of the normal are perpendicular to the traces DB, DC of plane (235). The normal passing through the point x,y,z on the surface, if x',y' and z', be the variable co-ordinates on the normal, the equations of its projections HS, and RL, are,

(237) $$\begin{cases} z' - z = a\,(x' - x), & \text{equation of HS,} \\ z' - z = b\,(y' - y), & \text{equation of RL.} \end{cases}$$

Recollecting that $p = \tan.DBX$, and $q = \tan.DCY$, we have by the usual relation of the tangents of perpendicular lines $a = -\frac{1}{p}$, and $b = -\frac{1}{q}$. These values change (237) to,

(238) $$\begin{cases} p\,(z' - z) + x' - x = o, \\ q\,(z' - z) + y' - y = o. \end{cases}$$

These are the equations of the normal line required, in which p and q are the partial differential coefficients from the surface (234). Substitute into (238) for p and q, their values deduced from the surface (234), and we have the equations of the normal line.

Ex. 1. Find the equations of the normal line to the sphere at the point x,y,z.

Here (234) becomes

(a) $$x^2 + y^2 + z^2 - r^2 = o,$$

from which we have,

$$\frac{dz}{dx} = -\frac{x}{z} = p \text{ and } \frac{dz}{dy} = -\frac{y}{z} = q,$$

and (238) becomes

(c) $$x' - x = \frac{x}{z}(z' - z) \text{ and } y' - y = \frac{y}{z}(z' - z).$$

Equations (c) are the equations of the normal required.

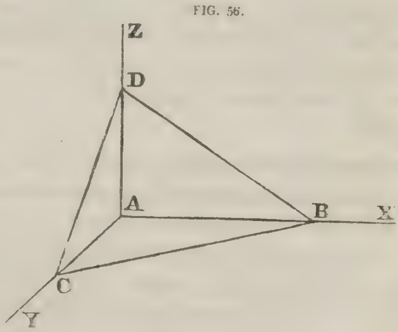
Ex. 2. Find the equation of the normal to the surfaces

$$x^2 + y^2 - 4mz = o, \qquad a^2y^2 + b^2x^2 + c^2r^2 - m^2n^2 = o.$$
$$a^2y^2 + b^2(x^2 + z^2) - a^2b^2 = o, \quad a^2y^2 - b^2(x^2 + z^2) + a^2b^2 = o.$$

PROPOSITION XLIX.

A tangent plane is drawn to a given curve surface, its trace on one of the co-ordinate planes makes with the co-ordinate axes in that plane, a triangle of given area, determine the point of tangency.

FIG. 56.



Let DBC be the tangent plane.

Suppose the plane on which the triangle of given area is made, be the plane of XY. Let the surface be,

(239) $$\varphi(x,y,z) = o.$$

The equation of the tangent plane is (235).

In (235), for $z' = o$, $x' = o$, we have,

(240) $$y' = \frac{px + qy - z}{q} = \text{AC}.$$

In (235), for $z' = o$, $y' = o$, we have,

(241) $$x' = \frac{px + qy - z}{p} = \text{AB}.$$

Put $v^2 =$ area CAB, and we have, from (240) and (241), by the rule for the area of a triangle,

(242) $$2pqv^2 = (px + qy - z)^2,$$

in which the partial differential coefficients p and q, are to be deduced from the equation of the surface (239).

Equations (239) and (242) are all the equations we have for determining the point of tangency. But as these two equations contain three unknown quantities, x, y, and z, there are an indefinite number of points, (x,y,z), through which the tangent plane may be drawn. If we first eliminate one of the variables, as z, between (239) and (242), and represent the resulting equation by

(243) $$\text{F}\,(x,y) = o,$$

and then eliminate another of the variables, as x, between (239) and (242), and represent the resulting equation by

(244) $$\psi(z,y) = o,$$

the two equations, (243) and (244), will be the projections on the planes of XY and ZY, respectively, of the line on surface (239), which is the locus of the point of tangency. There may, therefore, be an indefinite number of tangent planes to surface (239), each cutting off the proposed triangle, the points of tangency forming on surface (239), a curve whose projections are (243), (244). If one of the co-ordinates be given, as

(245) $$z = m,$$

we have three equations, (239), (242), (245), to find x, y, and z, which being determined by the solution of these three equations, make known the point of tangency.

If we put into (235) the values of p and q, taken from the surface (239), we may employ the resulting equation of the tangent plane in forming the equation (242).

Let the surface be the sphere with centre at the origin.

Here (239) becomes,

(a) $$x^2 + y^2 + z^2 - \text{R}^2 = o,$$

and the equation of the tangent plane to the sphere is,

(*b*) $$xx' + yy' + zz' = R^2.$$

Proceed with (*b*) as in (240), (241), and we have, for the area corresponding to (242),

(*c*) $$2v^2xy = R^4.$$

Equation (*c*) shows, without further elimination, that the projection on the plane of XY, of the locus of the point of tangency on the sphere, is a hyperbola, whose asymptotes are the axes of X and Y.

Eliminate x between (*a*) and (*c*), and we have a line of the fourth order, for the projection on ZY of the locus of tangency. If we suppose one co-ordinate z given, as in (245), we may find the co-ordinates of the point of tangency, by solving (245), (*a*), and (*c*), for x, y, and z.

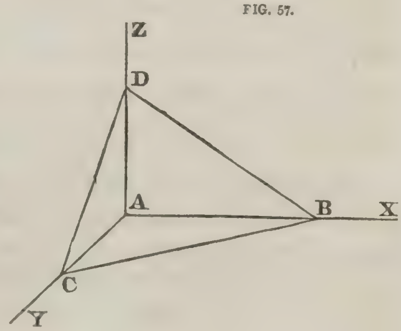
Ex. 2.—Determine the locus of such a point of tangency in the surfaces,

$$x^2 + y^2 - 4mx, = o, a^2 x^2 + a^2 y^2 + b^2 z^2 - a^2 b^2 = o.$$

PROPOSITION L.

A tangent plane is drawn to a given curve surface, its traces on two of the co-ordinate planes make with the axes triangles, each of whose areas is given, determine the point of tangency.

FIG. 57.



Let CAB and DAB be the triangles of given area.

Put u^2 = the area of DAB. Proceed as in last Proposition, to (242).

Then to find AD, put

$$x' = o, \quad y' = o,$$

in (235), and we have

(246) $$z' = z - (px + qy) = AD.$$

By means of (246), (241), we form the expression for the area DAB. Hence,

(247) $$2pu^2 = -(px + qy - z)^2.$$

From equations (247), (242), (239), we can find x, y, and z, the co-ordinates of the point of tangency required.

Ex. 1.—Let the surface be the sphere, find such a point of tangency.

Ex. 2.—Let the surface be the paraboloid or ellipsoid of revolution. Find such a point of tangency.

PROPOSITION LI.

Determine the distance from a given point on a curve surface to a given plane, measured on the normal to the surface.

Call P the point on the curve surface, and D the point where the normal meets the given plane. Let

(248) $$Ax' + By' + Cz' + D = o,$$

be the given plane, and

(249) $$\varphi(x,y,z) = o,$$

the given surface.

If N be the distance PD, the length required, we have from Analytical Geometry, (putting (x,y,z) for the co-ordinates of P, and (x',y',z') for the co-ordinates of D).

(250) $$N = \left((x' - x)^2 + (y' - y)^2 + (z' - z)^2\right)^{\frac{1}{2}},$$

The equations of the normal line are, by (238),

(251) $$\begin{cases} p(z' - z) + x' - x = o. \\ q(z' - z) + y' - y = o. \end{cases}$$

At the point where the normal line pierces the plane, the co-ordinates x', y', and z', are common to (248), (259) and (251). Find these co-ordinates from the three equations (248), (251), substitute into (250), and we have the length required.

Ex.—Find such a distance in the surfaces,

(M) $\begin{cases} x^2 + y^2 + z^2 - R^2 = o, & x^2 + y^2 - 4mz = o, \\ a^2(x^2 + y^2) + b^2z^2 - a^2b^2 = o, & x^2 + y^2 - m^2z^2 = o. \end{cases}$

PROPOSITION LII.

Determine the distance from a given point on a curve surface to each of the co-ordinate planes, measured on the normal to the surface.

Where the normal pierces the plane of XY, the ordinate z' is zero in (250) and (251), and these equations become

(252) $$N = \left((x' - x)^2 + (y' - y)^2 + z^2 \right)^{\frac{1}{2}},$$

(253) $$\begin{cases} -pz + x' - x = o, \\ -qz + y' - y = o. \end{cases}$$

Eliminate x', y', from (252), by means of (253), and we have for the distance required from the plane XY,

(254) $$N = z\,(p^2 + q^2 + 1)^{\frac{1}{2}}.$$

Proceeding in the same manner, we find for N′ the distance required from the plane ZY,

(255) $$N' = \frac{x}{p}\,(p^2 + q^2 + 1)^{\frac{1}{2}},$$

and for the distance N″ from ZX, we find,

(256) $$N'' = \frac{y}{q}\,(p^2 + q^2 + 1)^{\frac{1}{2}},$$

Ex.—Determine these distances in the surfaces (M), last Proposition.

PROPOSITION LIII.

A plane is drawn through a given point, and tangent to a given surface, determine the point of tangency.

Let (a,b,c) be the co-ordinates of the given point, and (x,y,z) the co-ordinates of the point of tangency.

The equation of the tangent plane is (235),

(257) $$z' - z = p\,(x' - x) + q\,(y' - y).$$

At the given point, the co-ordinates x', y', and z', of this plane become a, b, and c, respectively, and (257) is,

$$c - z = p(a - x) + q(b - y). \tag{258}$$

Let now the equation of the surface be

$$\varphi(x,y,z) = o. \tag{259}$$

At the point of tangency, the co-ordinates x,y,z are common to (258), (259), and these are the only two equations we have to determine the point of tangency. If we first eliminate one of the variables, as z, between (258) and (259), and represent the resulting equation by

$$\mathrm{F}(x,y) = o, \tag{260}$$

and then eliminate another of the variables as x, between (258) and (259), and represent the resulting equation by

$$\psi(z,y) = o, \tag{261}$$

the two equations (260), (261), will be the projections on the planes XY and ZY, of the line on surface (259), which is the locus of the point of tangency. There may therefore be an indefinite number of tangent planes to surface (259), each passing through the given point, and the points of tangency forming on surface (259) a curve whose projections are (260), (261), If one of the co-ordinates, as z, of the point of tangency be given, we can determine the other two from (259) and (258).

Ex. 1.—Let tne given surface be a sphere. Then (259) is,

$$x^2 + y^2 + z^2 - \mathrm{R}^2 = o, \text{ and (258) is,} \tag{a}$$

$$ax + by + cz - \mathrm{R}^2 = o, \tag{b}$$

Eliminate first z and then x between (a) and (b), and (260), (261) are ellipses.

Ex. 2.—Determine the locus of such a point of tangency in surfaces (M), Proposition LI.

PROPOSITION LIV.

A plane is drawn through two given points and touches a given surface, determine the point of tangency.

Let (a,b,c) and (d,e,f), be the two given points. Since the

tangent plane passes through (d,e,f), its equation for that point is,

(262) $$f - z = p\,(d - x) + p\,(e - y),$$

and we have, (258), (259), and (262), three equations to find x, y, and z, the co-ordinates of the point of tangency required.

Ex.—Determine such a point of tangency in surfaces (M), Proposition LI.

PROPOSITION LV.

A sphere passes through a given point, and touches a given surface, determine the locus of its centre.

Let (a,b,c) be the variable co-ordinates of the centre of the sphere, then the equation of the sphere is,

(263) $$(x - a)^2 + (y - b)^2 + (z - c)^2 = R^2.$$

Let (d,e,f) be the given point through which the sphere passes. For this point the equation of the sphere (263) becomes,

(264) $$(d - a)^2 + (e - b)^2 + (f - c)^2 = R^2.$$

Let the equation of the given surface be represented by

(265) $$\phi(x,y,z,) = o.$$

Let p and q, represent the partial differential coefficients of (263) and p' and q', the partial differential coefficients of (265).

At the point where (263) touches (265), if we suppose a plane tangent to (263) drawn, it will be tangent to (265); and since the partial differential coefficients of (263) and (265), at the point of tangency, express the tangents of the angles which the traces of the same tangent plane make with the axes of x and y, we have,

(266) $$p = p', \text{ and}$$

(267) $$q = q'.$$

At the point of tangency the co-ordinates x, y and z, are common to the four equations (263), (265), (266), (267), eliminate these three co-ordinates between these equations. Represent the resulting equation by

(268) $$\phi(a,b,c,\ R) = o,$$

and we have, (268) and (264) to eliminate the variable radius R. The result of this elimination, is an equation containing constants

and the co-ordinates (a,b,c) of the centre of the sphere. Represent this equation by

(269) $$\psi(a,b,c) = o,$$

and we have the equation of the locus required.

We might with more brevity say that (269) is obtained by eliminating the four quantities x,y,z, and R, between the five equations (263) (264), (265), (266), and (267).

Ex. 1.—Let the given surface be the sphere whose origin is at the centre, find the locus of the centre of (263),

Here (265) becomes,

(*a*) $$x^2 + y^2 + z^2 = s^2,$$

equating the partial differential cofficients of (*a*) and (263), we have for (266) and (267), the equations,

(*b*) $$cx = az, \quad \text{and } cy = bz.$$

Eliminate x,y,z, and R, between the five equations (263), (*a*), (*b*), and (264), and we have, for the locus required,

(*c*) $$a^2\left(1 - \frac{d^2}{s^2}\right) + b^2 + c^2 - \frac{ad}{s^2}(s^2 - d^2) - \frac{(s^2 - d^2)^2}{4s^2} = o.$$

When d is zero (*c*) is a sphere, when d is less than s, (*c*) is an ellipsoid of revolution, when d is greater than s, (*c*) is a hyperboloid of revolution, and when d equals s, (*c*) is a point.

Ex. 2.—Let the surface (265) be a plane passing through the origin, find the locus of the centre of (263) passing through a given point and touching this plane.

PROPOSITION LVI.

A sphere touches two given surfaces, determine the locus of its centre.

Let (a,b,c) be the co-ordinates of the centre. Let

(269) $$\phi(x',y',z') = o,$$

be one surface, and

(270) $$\psi(x'',y'',z'') = o, \text{ the other.}$$

Let the equation of the sphere for the point where it touches (269), be

(271) $$(x' - a)^2 + (y' - b)^2 + (z' - c)^2 = R^2,$$
and for the point where it touches (270),
(272) $$(x'' - a)^2 + (y'' - b)^2 + (z'' - c)^2 = R^2.$$

Let p and q be the differential coefficients of (271),

P and Q	Do.	of (269),
p' and q'	Do.	of (272),
P$'$ and Q$'$	Do.	of (270).

Then as in the previous proposition, at (266), and (267), we have the equations,
(273) $$p = P \quad \text{and } q = Q,$$
(274) $$p' = P' \quad \text{and } q' = Q'.$$

In (273), p,q,P and Q are functions of x',y',z', and in (274), p',q', P$'$ and Q$'$, are functions of x'',y'', and z''.

From the four equations (269), (271), and (273), eliminate the three co-ordinates x',y', and z'. The resulting equation will contain a,b,c, and R, and may be represented by
(275) $$F(a,b,c,R) = o.$$

From the four equations (270), (272), and (274), eliminate x'', y'', z''. The resulting equation may be represented by
(276) $$f(a,b,c,R) = o.$$

Between (275) and (276), eliminate the variable radius R. The resulting equation may be represented by
(277) $$\phi(a,b,c) = o,$$
which is the locus required.

Ex.—As an example, let (270) be a plane coinciding with the plane of ZY, and (269) a sphere whose equation is,
(a) $$(x' - m)^2 + y'^2 + z'^2 = m^2,$$
its centre being on the axis of x, and its surface passing through the origin. We have at once for (276),
(b) $$a = R,$$
and equating the partial differential coefficients of (a) and (271) we have,
(c) $$cy' = bz, \qquad c(x' - m) = (a - m)z'.$$

Eliminate x,$'y$,$'z'$, andR, between the five equations (271), (a), (c), and (b), and we have,
(d) $$b^2 + c^2 = 4ma.$$

Which shows that the locus is the surface of a paraboloid of revolution passing through the origin.

PROPOSITION LVII.

Determine the differential of an implicit equation.

We have already defined an implicit equation to be one that is not solved for any particular variable, and we have deduced the differential coefficient from implicit equations, where they were of a simple form. The only object of this propoposition is, to determine a convenient process for differentiating and obtaining the differential coefficients when the equation is complicated.

Suppose we have the equations of curves,

$$\text{(N)} \qquad \begin{cases} y^2 - px = o & (a) \\ x^2 + y^2 - r^2 = o & (b) \\ a^2\,y^2 + b^2x^2 - a^2\,b^2 = o & (c). \end{cases}$$

We observed (in Proposition II.) that all such equations may be represented by the form,

$$(278) \qquad \varphi(x,y) = o.$$

We may also represent equations (N) by the still simpler form,

$$(279) \qquad u = o,$$

in which u is considered as containing x and y, or as put for all such equations as (N).

If we differentiate the second of (N) or (b) for y constant, we have,

$$(d) \qquad 2xdx = o,$$

which may be called the partial differential of (b).

In like manner we may take the partial differential of (279), and write it,

$$(e) \qquad \frac{dudx}{dx} = o,$$

where by introducing dx into the numerator and denominator, we signify that (279) has been differentiated for x variable.

Again differentiate (b) for x constant, and we have,

$$(f) \qquad 2y\,dy = o,$$

which is the partial differential of (b) in respect of y.

In like manner the partial differential of (279) is,

$$(g) \qquad \frac{du\,dy}{dy} = o,$$

where dy is introduced in the same manner, and for the same purpose as dx in (e).

If we add (d) and (f), we have,

$$(h) \qquad 2x\,dx + 2y\,dy = o,$$

which is the total differential of (b). In like manner add (e) and (g), and we have for the sum of the partial differentials,

$$(280) \qquad \frac{du\,dx}{dx} + \frac{du\,dy}{dy} = o,$$

which is the total differential of (279).

Divide (280) by dx, and we have,

$$(281) \qquad \frac{du}{dx} + \frac{du}{dy} \cdot \frac{dy}{dx} = o,$$

from which we get

$$(282) \qquad \frac{dy}{dx} = -\frac{du}{dx} \div \frac{du}{dy} = p.$$

This is the general form of the differential coefficient of (279).

Since we have assumed $u = o$, as the representative of (278), we may, in using the form (282), for obtaining the differential coefficient, assume u equal to the equation of the curve.

Thus in the case of the circle, put u equal to the second of (N), that is,

$$(k) \qquad u = x^2 + y^2 - r^2 = o.$$

Differentiate (k), first for y constant, and we have, after dividing by dx,

$$(l) \qquad \frac{du}{dx} = 2x.$$

Differentiate (k) next for x constant, and we have,

$$(m) \qquad \frac{du}{dy} = 2y.$$

Substitute (l) and (m) into (282), and we have,

$$(n) \qquad \frac{dy}{dx} = -\frac{x}{y},$$

which is obviously the same that would be deduced from (h), by the usual rules of Algebra.

As an application of the form (282), let us take Proposition III., which proposed *to determine the equation of a tangent line to a plane curve.*

In (65), we found the equation of the tangent line to be,

(282a) $$y' - y = p\,(x' - x).$$

For p in this, put its value in (282), and we have for the equation of the tangent line to a plane curve,

(282b) $$\frac{du}{dy}(y' - y) + \frac{du}{dx}(x' - x) = o.$$

Put into this the values of $\frac{du}{dy}$ and $\frac{du}{dx}$, deduced from the equation of the curve, $u = o$, and we have the equation of the tangent line required.

Ex. 1.—Determine the equation of the tangent line to a parabola.

Put u equal to the equation of the parabola, and we have,

(2c) $$u = y^2 - 4mx = o.$$

Differentiate this for x constant, and we have,

(2d) $$\frac{du}{dy} = 2y.$$

Differentiate (2c) for y constant, and we have,

(2e) $$\frac{du}{dx} = -\ 4m.$$

Put into (282b) the values of the differential coefficients at (2d) and (2e), and we have for the equation of the tangent line to the parabola,

(2f) $$2y\,(y' - y) - 4m\,(x' - x),$$

or adding the double of (2c) to (2f), we have,

(2g) $$yy' - 2m\,(x' + x) = o.$$

Determine in this manner the equation of the tangent line to the curves,

$$a^2\,y^2 + b^2\,x^2 - a^2\,b^2 = o, \qquad xy - y^2 + ay - bx = o.$$

Let us next take an implicit equation of three variables. Every such equation is the equation of a surface, and may be represented by the form,

(283) $$\varphi(x,y,z) = o.$$

We may also represent any surface by the still simpler form,

(284) $$u = o,$$

in which u is supposed to contain x, y, and z, and is a representative of (283), or of any equation of three variables. As (284) contains three variables, it will have three partial differentials. Using the same notation as in (e), and (g), we have for the form of the differential of (284),

(285) $$\frac{du\,dx}{dx} + \frac{du\,dy}{dy} + \frac{du\,dz}{dz} = o,$$

the sum of the three partial differentials.

If we suppose y constant, in (284), (which makes the second term of (285) zero), we have for the partial differential of (284), divided by dx,

(285a) $$\frac{du}{dx} + \frac{du}{dz} \cdot \frac{dz}{dx} = o.$$

By making x constant in (284), (which makes the first term of (285) zero), we have, for the partial differential of (284), divided by dy,

(285b) $$\frac{du}{dy} + \frac{du}{dz} \cdot \frac{dz}{dy} = o.$$

From (285a) we have,

(286) $$\frac{dz}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dz}} = p,$$

and from (285b) we have,

(287) $$\frac{dz}{dy} = -\frac{\frac{du}{dy}}{\frac{du}{dz}} = q.$$

To obtain the partial differential coefficients (286), (287), we may assume u equal the equation of the surface, and deduce from this equation the several differential coefficients that enter (286), (287).

As an example, let the surface be the sphere. Then putting u equal to the equation of the sphere, we have,

$$(r) \qquad u = x^2 + y^2 + z^2 - r^2 = o,$$

and differentiating this first for x variable, next for y variable, and finally for z variable, we have,

$$(s) \qquad \frac{du}{dx} = 2x, \qquad \frac{du}{dy} = 2y, \qquad \frac{du}{dz} = 2z.$$

Substitute the several values of (s) into (286), (287), and we have,

$$(t) \qquad \frac{dz}{dx} = -\frac{x}{z}, \text{ and } \qquad \frac{dz}{dy} = -\frac{y}{r},$$

which are the partial differential coefficients of (r).

As an example of the application of the forms (286) and (287), let it be proposed *to determine the equation of a tangent plane to a curve surface.*

If the differential coefficients (286), (287), be substituted into the equation of the tangent plane (229), that equation becomes,

$$(288) \qquad \frac{du}{dz}(z' - z) + \frac{du}{dy}(y' - y) + \frac{du}{dx}(x' - x) = o,$$

which is the equation of the tangent plane to any surface (284).

To apply (288) to find the tangent plane to the sphere, substitute into (288) the values of the differential coefficients (s), and we have, after adding the double of (r) to the result,

$$(v) \qquad xx' + yy' + zz' - r^2 = o,$$

(as in Proposition XLV., (c),).

In like manner, find the equation of the tangent plane to the surfaces,

$$(\mathrm{P}) \qquad \left\{ \begin{array}{l} x^2 + y^2 - m^2 z^2 = o, \\ m^2 (y - bz)^2 + n^2 (x - az)^2 - m^2 n^2 = o, \\ (x - a)^2 + (y - b)^2 + (z - c)^2 - \mathrm{R}^2 = o, \end{array} \right.$$

of which the first is the cone, the second the oblique elliptic cylinder, and the last the sphere.

PROPOSITION LVIII.

Determine the angles which the normal line at a given point on a curve surface makes with the co-ordinate axes.

If the equations of the projections of a line in space be

(289) $y = bz + m,$ and $x = az + n,$

and θ, θ', and θ'' be the angles which this line makes with the coordinate axes X, Y, and Z, respectively, it is shown in all books on Analytical Geometry, that,

(290) $\cos.\theta = \dfrac{a}{(a^2+b^2+1)^{\frac{1}{2}}}.\quad \cos.\theta' = \dfrac{b}{(a^2+b^2+1)^{\frac{1}{2}}},$

$$\text{and } \cos.\theta'' = \frac{1}{(a^2+b^2+1)^{\frac{1}{2}}}.$$

If the line in space be the normal, the equations of its projections are by (239),

(291) $x' - x = -p\,(z' - z),$ and $y' - y = q(z' - z).$

In (291), $-p$ and $-q$ take the place of a and b in (289). Hence for a and b in (290) take the values $-p$ and $-q$, and we have the angles required in terms of the differential coefficients of the given point on the surface. If for p and q we take their values in (286), (287), the angles (290) become,

(292) $\cos.\theta = v\dfrac{du}{dx}, \cos.\theta' = v\dfrac{du}{dy}, \quad \cos.\theta'' = v\dfrac{du}{dz},$

where we put for brevity,

(293) $\dfrac{1}{v} = \left\{\left(\dfrac{du}{dx}\right)^2 + \left(\dfrac{du}{dy}\right)^2 + \left(\dfrac{du}{dz}\right)^2\right\}^{\frac{1}{2}}.$

Ex. 1.—Suppose the surface be a sphere, find the angles the normal to the sphere makes with the axes.

Substitute the values (s) last Proposition into (293), (292), and putting the value of v from (293) into (292), we have for the angles required,

(w) $\cos.\theta = \dfrac{x}{r}, \quad \cos.\theta' = \dfrac{y}{r}, \quad \cos.\theta'' = \dfrac{z}{r}.$

Ex. 2.—Find these angles in the surfaces

$$x^2 + y^2 - 4mz = o, \qquad x^2 + y^2 - m^2 z^2 = o.$$

CONSECUTIVE SURFACES.

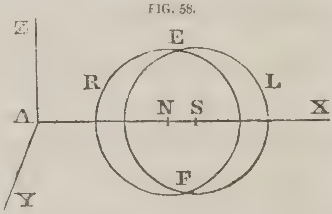
The nine following Propositions will be devoted to the discussion of Consecutive Surfaces; and though each Proposition may of itself fail of that generality which is possessed by those hitherto discussed, yet taken together, they form an interesting part of the Geometry of Surfaces, and exhibit much of the power of the Differential Calculus.

PROPOSITION LIX.

The centre of a sphere moves along a given line, determine the locus of the intersection of the sphere with its consecutive sphere.

We will illustrate this by taking a few particular cases, in which it will be seen that the line on which the centre moves, determines the nature of the locus of the intersections.

FIG. 58.



Ex. A.

Suppose the line on which the centre moves be the axis of X.

If N be the centre of the sphere in one position, and AN $= a$, then

(294) $$(x - a)^2 + y^2 + z^2 = \mathrm{R}^2,$$

is the equation of the sphere. For the equation of the consecutive sphere, increase AN by NS $= da$, and we have from (294),

(295) $$(x - a - da)^2 + y^2 + z^2 = \mathrm{R}^2.$$

Where the two spheres (294), (295), intersect, the co-ordinates x,y, and z, are common. If we subtract (294) from (295), the co-ordinates x,y, and z, of the resulting equation will appertain to the line of intersection of the two spheres. But subtracting (294) from (295), and making da indefinitely small or zero, is obviously differentiating (294) for a variable. Differentiate, therefore, (294), for a variable, and we have,

(296) $$-2(x-a)\,da = o.$$

In (296) the co-ordinates of the surface, (in this case x) appertain to the intersection of (294) with its consecutive surface. If now we eliminate a between (294) and (296), the resulting equation which is,

(297) $$y^2 + z^2 = R^2,$$

is the equation of the surface, which is the locus of the intersection of the sphere in any position with its consecutive sphere. Equation (297) is the equation of a cylinder perpendicular to the plane ZY, and whose axis coincides with the axis of X.

Ex. B.

Suppose the line, on which the centre moves, pass through the origin, and lie in the plane XY. Let its equation be,

(298) $$\beta = ma.$$

When the centre of the sphere is at any point (a,β) of the line (298), its equation is

(299) $$(x-a)^2 + (y-\beta)^2 + z^2 = R^2.$$

By virtue of (298), (299) becomes,

(300) $$(x-a)^2 + (y-ma)^2 + z^2 = R^2.$$

In this, the parameter a determines the position of the sphere.

Differentiate (300) for a variable, and we have,

(301) $$-2\,(x-a)\,da - 2m\,(y-ma)\,da = o.$$

The co-ordinates of the surface in (300) and (301) are common on the line of intersection of (300) with its consecutive sphere.

Eliminate the parameter a between (300) and (301), and we have,

(302) $$(y-ma)^2 + z^2 = R^2.$$

This is the equation of the surface, which is the locus of the intersection of the sphere, in any position, with its consecutive sphere.

If $m = o$, the line (298) coincides with the axis of X, and (302) becomes (297).

Ex. C.

If instead of a straight line, (298), the centre of the sphere moves along any curve in the plane XY, then representing that curve by

(303) $$\beta = \phi a,$$

the equation of the sphere (300) becomes,

(304) $$(x-a)^2 + (y-\phi a)^2 + z^2 = R^2.$$

Eliminate a between (304) and its differential, for a variable. The result is the equation of the surface, which is the locus of the intersections of (304) with its consecutive surface.

If (303) be a circle whose radius is r, and centre at the origin, the surface which is the locus of the intersection of the sphere, in any position with its consecutive sphere, is

$$(305) \qquad \left\{ r \pm (x^2 + y^2)^{\frac{1}{2}} \right\}^2 + z^2 = R^2.$$

The line of intersection of a surface with its consecutive surface, is called, "*The Characteristic*," a name proposed by M. Monge.

Corrol. 1st.—The characteristic, when the surface is a sphere, (as in the previous examples), is a great circle of the sphere whose plane is perpendicular to the line on which the centre of the sphere moves.

For let the line be in the plane of XY, and let (303) be its equation, and (299) the equation of the sphere in any position. Differentiate (299) for a and β variable, and we have,

$$(306) \qquad y - \beta = -\frac{da}{d\beta}(x - a).$$

The co-ordinates x and y, of this equation, are co-ordinates of the characteristic. Since z is absent from it, (306) may be regarded as the projection of the characteristic on the plane of XY. But (306) is the equation of a straight line normal to the curve (303), [see (67)], at the point (a,β). But the point (a,β) is the centre of the sphere (299). Hence the plane of the characteristic passes through the centre of the sphere, and is perpendicular to the line (303), on which the centre moves. The characteristic is therefore a great circle of the sphere.

From this it is evident, that the surface which is the locus of these characteristics, is tangent to the sphere in every position of it.

For let

$$(307) \qquad \varphi(x,y,z) = o,$$

represent the surface obtained by eliminating a between (304) and its differential, for a variable, (307) touches the sphere in all its positions, the line of contact evidently being the characteristic. As a particular case of this, (279) is a cylinder circumscribing (294).

In like manner, (305) is the surface tangent to the sphere in every position.

Corrol. 2d.—Let, in general,

(308) $$u = o,$$

be the equation of a surface containing the variable parameter β, and the cordinates x, y, and z. If we differentiate (308) for x, y, and z constant, and β variable, the differential, which being divided by $d\beta$, may be represented by

(309) $$\frac{du}{d\beta} = o,$$

will contain the co-ordinates x, y, and z, which are common to (308), and (309), on the characteristic, or line of intersection of (308) with its consecutive surface. If β be eliminated between (308), (309), the resulting equation may be represented by

(310) $$u' = o.$$

This equation contains x,y,z, and is the locus of the characteristics of (308).

Corrol. 3d.—The locus of the characteristics of (308), viz. (310), is tangent to (308).

For differentiate (308) for β constant, and we have for the partial differential coefficients, [as in (285a), (285b)],

(311) $$\frac{du}{dx} + \frac{du}{dz}p = o, \text{ and } \quad \frac{du}{dy} + \frac{du}{dz}q = o.$$

Again, differentiate (308) for x, y, z, and β variable, and we have for the partial differential coefficients,

(312) $$\frac{du}{dx}+\frac{du}{dz}p+\frac{du}{d\beta}\cdot\frac{d\beta}{dx} = o, \text{ and } \quad \frac{du}{dy}+\frac{du}{dz}q+\frac{du}{d\beta}\cdot\frac{d\beta}{dy} = o.$$

The x, y, and z of (311) and (312), are common on any characteristic, or on the locus (310). But for any characteristic, (309) exists, and (309) renders (311) and (312) identical. Hence p and q being the same in (311) and (312), the tangent plane to (308), at any point common to (308) and (310), is a tangent plane to (310). The surfaces (308,) (310), are therefore tangent to each other.

The surface (310) is generally called, *The Envelope* of the surface (308).

Corrollary 2d shows the mode, in general, of obtaining the envelope of a surface, and Corollary 3d proves the general proposition,

that the envelope is tangent to the surface whose consecutive intersections, or characteristics, form the envelope.

As a further elucidation of these principles, take the following Proposition.

PROPOSITION LX.

The centre of a sphere moves along a given line, its radius varies as a given function of the distance of the centre from a given point, determine the surface which touches and envelopes the sphere in every position.

Instead of writing down the general form of the solution to this Proposition, we will particularise one or two examples.

Ex. A.

Let the line on which the centre of the sphere moves be the axis of X, and let the radius vary as the distance of the centre from the origin.

Let a be the distance of the centre of the sphere in any position from the origin, and since the radius varies as this distance, we may put $R = ma$, where m is any given constant, and the equation of the sphere is,

(313) $$(x - a)^2 + y^2 + z^2 = m^2a^2.$$

Differentiate this for a variable.

(314) $$\therefore -(x - a) = m^2a.$$

Eliminate a between (313) and (314), and we have,

(315) $$y^2 + z^2 = \frac{x^2m^2}{1-m^2},$$

which is the surface required. Equation (315) is the equation of a cone.

Ex. B.

Let the centre of the sphere move on the axis of X, and its radius vary as the square root of the distance from the centre to the origin.

Here put

$$R = (ma)^{\frac{1}{2}},$$

which expresses the given law of variation of the radius, and the equation of the sphere in any position is,

(316) $$(x - a)^2 + y^2 + z^2 = ma.$$

Eliminate a between (316) and its differential for a variable, and we have,

(317) $$y^2 + z^2 = mx + \frac{m^2}{4},$$

the enveloping surface, which is a paraboloid of revolution.

Ex. C.

Let the line on which the centre of the sphere moves be any curve in the plane of XY, and let the radius vary as any function of the abscissa of the centre.

Let (a,β) be the co-ordinates of the centre, and

(318) $$\beta = \varphi a,$$

be the curve on which the centre moves.

Let $R = fa$ denote the function which the radius is of the abscissa of the centre. Then the equation of the sphere in any position is,

(319) $$(x - a)^2 + (y - \beta)^2 + z^2 = (fa)^2.$$

Substitute into (319) the value of β in (318), and we have for the equation of the sphere, involving the single parameter a,

(320) $$(x - a)^2 + (y - \varphi a)^2 + z^2 = (fa)^2.$$

For any particular values of fa and φa, we can eliminate a between (320) and its differential for a variable. The result is the envelope of (319).

Ex. D.

If the radius in the preceding example varied as a given function of the co-ordinates of the centre, then denoting this law of variation by $R = \psi(a,\beta)$, we have for the equation of the sphere,

(321) $$(x - a)^2 + (y - \beta)^2 + z^2 = \left(\psi(a,\beta)\right)^2.$$

Eliminate β from (321) by the relation (318), proceed as directed in regard to (320), and we get the envelope required.

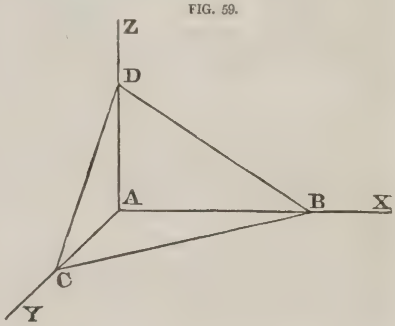
By differentiating (321) (which includes all the previous examples), for a and β variable, we could show as in the previous proposition, that the characteristic is a great circle of the sphere whose

plane is perpendicular to the line on which the centre of the sphere moves. This fact would show, without the proof in Corrol. 3d, last Proposition, that the locus of these characteristics is a surface touching and enveloping the sphere in every position.

PROPOSITION LXI.

Through a given point a plane is drawn, cutting off on the plane of XY a triangle of given area, determine the surface to which the plane is always tangent.

FIG. 59.



For simplicity, let D be the given point on the axis of Z, and let

(322) $z = -mx - ny + p$,

be the plane in one position CDB, cutting off the given triangle ABC. Put $a^2 =$ area ABC, which is given.

For x and y zero in (322), we have,

$$(A)\quad \begin{cases} z = p = \text{AD}. & \text{For } x \text{ and } z \text{ zero in the same, we have,} \\ y = \dfrac{p}{n} = \text{AC}. & \text{For } y \text{ and } z \text{ zero, we have,} \\ x = \dfrac{p}{m} = \text{AB}. & \text{For the area of the triangle ABC, we have,} \end{cases}$$

$$(323)\qquad a^2 = \frac{p^2}{2mn}.$$

From which we have,

$$m = \frac{p^2}{2a^2n}.$$

Substitute this value into (322), and we have,

$$(324)\qquad z = -\frac{p^2}{2a^2n}x - ny + p.$$

This is the equation of the plane involving the arbitrary parameter n. Differentiate (324) for n variable, and we have,

$$o = \frac{p^2 x dn}{2a^2n^2} - ydn. \tag{325}$$

The co-ordinates x and y in (325) appertain to the characteristic or line of intersection of (324) with its consecutive plane. Eliminate n between (324) and (325). The result is,

$$a^2 (z - p)^2 - p^2 \, xy = o. \tag{326}$$

This is the surface which is the locus of all the characteristics of (324) and according to Corrol. 3d, Proposition LIX, touches the plane (324) in every position.

It is evident that the characteristic will in this case be a straight line, whose projection on the plane XY is, [from (325)],

$$p^2x - 2a^2ny = o. \tag{327}$$

Equation (327) designates a straight line through the origin. Hence, all the characteristics intersect the axis of Z, and pass through D.

The obvious properties of surface (326) are, that all sections parallel to ZX or ZY are parabolas, and all sections parallel to the plane XY, are hyperbolas, whose asymptotes are the intersections of the plane of the section with the planes ZX and ZY.

The nature of this surface might have been inferred from (187).

PROPOSITION LXII.

Through a given point on the axis of Z, a plane is drawn, whose traces make with the co-ordinate axes, triangles on the planes ZX and ZY, the sum of whose areas is given. Determine the surface to which this plane is tangent.

Let the given point be at D, on the axis of Z, and let (322) be the plane CDB, in one position.

Let s^2 = the sum of the areas DAB + CAD. Using the lines, (A), last Proposition, we have for the sum of these areas,

$$s^2 = \frac{p^2}{2m} + \frac{p^2}{2n}. \tag{328}$$

By means of (328), eliminate m from the plane (322), and proceed as was done with (324). The result is,

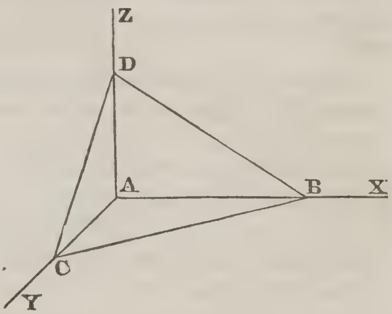
(329) $$2s^2(z - p) = - p^2 (x^{\frac{1}{2}} + y^{\frac{1}{2}})^2.$$

This is the surface required. Its other properties are readily deduced from its equation.

PROPOSITION LXIII.

A plane is drawn, cutting off, with the co-ordinate planes, a given pyramid. Determine the surface to which the plane is always tangent.

FIG. 60



Let DCB be the plane in one position, and (322) its equation.

Let s^3 = the solidity of the pyramid formed by plane (322) with the co-ordinate planes. Using the values (A), in Proposition LXI., we have for the volume of the pyramid,

(330) $$s^3 = \frac{p^3}{6mn}.$$

Substitute the value of m from (330) into (322), and we have for the equation of the plane DBC, which cuts off the given pyramid,

(331) $$z = - \frac{p^3 x}{6s^3 n} - ny + p.$$

As the plane is not limited to pass through a given point, the parameters n and p are both variable in (331).

It is evident that n and p are independent variables; for n may vary, giving a plane which fulfils the conditions of the Proposition, while p remains constant; and p may vary, giving a plane, which fulfils the same conditions, while n remains constant.

Differentiate, therefore, (331) for n variable, and we have, [after freeing from fractions],

(332) $$o = p^3xdn - 6s^3n^2y\,dn.$$

Again, differentiate (331) for p variable, and we have, [after freeing from fractions],

(333) $$o = -xp^2\,dp + 2s^3n\,dp.$$

The co-ordinates x, y, and z, in (331) and (332), are common on the characteristic, or line of intersection of (331), with its consecutive plane made by n variable.

The co-ordinates x, y, and z, in (331) and (333), are common on the characteristic, or line of intersection of (331), with its consecutive plane made by p variable.

Hence at the point of intersection of these two characteristics, the co-ordinates x,y,z, are common to (331), (332), and (333).

If, therefore, n and p be eliminated between these three equations, the co-ordinates of the resulting equation will appertain to the point of intersection of three of the planes consecutive two and two. The locus of this point is therefore the surface required.

Its equation, found by eliminating n and p between (331), (332), (333), is

(334) $$xyz = \frac{2}{9}s^3.$$

The obvious and remarkable properties of this surface are,

That it is asymptotical to each of the co-ordinate planes: That every section parallel to any of the co-ordinate planes is a hyperbola, whose asymptotes are the traces of the plane of the section, on the co-ordinate planes.

By a similar process, would be solved the two following problems.

A.

A plane is drawn cutting the co-ordinate planes so that the sum of the areas of the triangles formed by its traces with the co-ordinate axes is a constant quantity. Determine the surface to which the cutting plane is tangent.

B.

A plane is drawn, cutting the co-ordinate planes so as to form with its traces a given triangle CBD. Determine the surface to which the plane is tangent.

The algebraic detail of these Propositions presents some difficulty

PROPOSITION LXIV.

. Determine the relation between the parameters of a plane which passes through a given point, and touches a given surface.

Let the given point be at the origin. Let the equation of the plane be,

(335) $$z = mx + ny,$$

and the equation of the surface,

(336) $$\phi(x,y,z) = o.$$

It is required to determine the relation between the parameters m and n, when (335) touches (336).

At the point of tangency, the co-ordinates are common in (335) and (336). If p and q be the partial differential coefficients of (335), and P and Q of (336), at the point of tangency, we have, evidently,

(337) $$p = \text{P}, \quad \text{and} \quad q = \text{Q}.$$

At the point of tangency, the co-ordinates x, y, and z are common in the four equations, (335), (336), (337). Eliminate these co-ordinates between these equations. The resulting equation will contain the parameters m and n. Solving it for m, we have,

(338) $$m = \phi n,$$

which is the relation required.

Cor.—If we substitute (338) into (335), we have,

(339) $$z = \phi nx + ny,$$

which is the equation of a plane tangent to surface (336), involving only one parameter, n.

If we eliminate the parameter n from (339), by the principles of consecutive surfaces, the resulting equation, which may be represented by

(340) $$\text{F}\,(x,y,z) = o,$$

will be the envelope, or locus of the characteristics of (339).

If we define a *Developable Surface* to be one, whose tangent plane, at any point, coincides with the surface along a straight line, (340) is a developable surface.

As a particular example, let the surface (336) be a sphere whose equation is,

(*a*) $$(x - a)^2 + y^2 + z^2 = R^2.$$

The relation between the parameters m and n, of the tangent plane (335), is found, by the foregoing process, to be

(*b*) $$m = R\left(\frac{1 + n^2}{a^2 - R^2}\right)^{\frac{1}{2}}.$$

Substitute this value of m into (335), and we have the tangent plane involving only one parameter n.. Eliminate the parameter n from this tangent plane, by the principle of Consecutive Surfaces, and we have,

(*c*) $$z^2 + y^2 = \frac{R^2}{a^2 - R^2}x^2,$$

which is the developable surface formed by the characteristics of (335).

Equation (*c*) shows this developable surface to be a cone.

If instead of taking the origin to be the point through which the tangent plane (335) passes, we were to take any other point whose co-ordinates are (a,b,c), then instead of (335), we would have,

(340*a*) $$z - c = m(x - a) + n(y - b),$$

for the tangent plane, and the solution would be as before.

PROPOSITION LXV.

Determine the relation between the parameters of a plane tangent to two given surfaces.

Let the equation of one surface be,

(341) $$\phi(x',y',z') = o,$$

and the equation of the other,

(342) $$\psi(x'',y'',z'') = o.$$

Let the equation of the plane touching these two surfaces be,

(343) $$z = mx + ny + B.$$

It is required to determine the relation between the parameters m n and B, when (343) touches (341) and (342).

At the point where (343) touches (341), the co-ordinates are com mon to both equations, and (343) may be written,

(344) $$z' = mx' + ny' + B.$$

At the point where (343) touches (342), (343) may be written,

(345) $$z'' = mx'' + ny'' + \text{B}.$$

If p and q be the partial differential coefficients of (344),
If p' and q' " " " (345),
If P and Q " " " (341),
If P′ and Q′ " " " (342),

then we have, as at (266),

(346) $$p = \text{P}, \qquad q = \text{Q},$$

and also,

(347) $$p' = \text{P}', q' = \text{Q}'.$$

In (346) the partial differential coefficients are functions of x' y' and z', and in (347) they are functions of x'' y'' and z''.

Eliminate x',y',z' from the four equations (341), (344), (346).

The resulting equation will contain m,n, and B, and may be represented by

(348) $$\text{F}\,(m,n,\text{B}) = o.$$

Eliminate x'' y'' z'', from (342), (345), (347). Represent the resulting equation by

(349) $$f(n,m,\text{B}) = o.$$

Solve (348) and (349) for two of the parameters, as B and n. These solutions may be represented by

(350) $$\text{B} = \phi m, \quad \text{and} \quad n = \pi m,$$

which are the relations required.

Corr. If we substitute from (350) into (343), the values of B and n, we have,

(351) $$z = mx + \pi m y + \phi m.$$

This is the equation of a plane tangent to the two surfaces (341) (342) and involving the parameter m.

By putting

$$\frac{d.\pi m}{dm} = \pi' m, \quad \text{and} \quad \frac{d.\phi m}{dm} = \phi' m,$$

the differential of (351) for m variable, divided by dm, may be written

(352) $$x + \pi' m\, y + \phi' m = o.$$

If m be eliminated between (351) and (352) the resulting equation will be the developable surface, which touches the two surfaces

(341) and (342), and is the envelope, or locus of the characteristics of (351).

We may represent this resulting equation by

(353) $$F\,(x,y,z) = o.$$

As a particular example, let the surfaces (341), (342), be the two spheres,

(*d*) $$\begin{cases} x'^2 + y'^2 + z'^2 - R'^2 = o, \\ (x'' - a)^2 + (y'' - \beta)^2 + (z'' - c)^2 - R''^2 = o. \end{cases}$$

In this case equations (346) become

(*e*) $$m = -\frac{x'}{z'}, \quad n = -\frac{y'}{z'},$$

and equations (347) become

(*f*) $$m = -\frac{x'' - a}{z'' - c}, \quad n = -\frac{y'' - \beta}{z'' - c}.$$

Eliminate x', y', and z' between (344), (*e*) and the first of (*d*), and we have,

(*g*) $$R'\,(m^2 + n^2 + 1)^{\frac{1}{2}} - B = o.$$

Eliminate x'', y'', and z'' between (345), (*f*) and the second of (*d*), and we have,

(*h*) $$R''\,(m^2 + n^2 + 1)^{\frac{1}{2}} - B - ma - n\beta + c = o.$$

Equations (*g*) and (*h*) are what (348) and (349) become in this case. Solve (*g*) and (*h*) for B and n, and we have the particular values of (350), from which the plane (351) is determined.

PROPOSITION LXVI.

Determine the locus of the intersection of consecutive characteristics.

In Proposition LXV, the surface (353) which touches the two surfaces (341), (342) may be considered as made up of the intersections [i. e. the characteristics] of plane (351) with its consecutive plane. We may regard (351) as the generating surface whose consecutive intersections give the characteristics which make up the enveloping surface (353). It is obvious that this generating surface (351) in each position contains two characteristics. If these two

characteristics intersect, their point of intersection is on three consecutive generating surfaces.

The co-ordinates in (352) appertain to one characteristic of (351). As (352) does not contain z it may be regarded as the projection on XY of the characteristic.

By the theory of consecutive lines if we differentiate (352) for m variable, the co-ordinates of the differential will appertain to the point of intersection of (352) with its consecutive characteristic. [Proposition XXXVII.] Putting, for brevity,

$$\frac{d.\pi'm}{dm} = \pi''m, \quad \text{and} \quad \frac{d.\phi'm}{dm} = \phi''m,$$

the differential of (352) for m variable may be written,

(354) $$\pi''m\,y + \phi''m = o.$$

If m be eliminated between (352) and (354), the resulting equation will contain x and y, and may be represented by

(355) $$\mathrm{F}\,(x,y) = o.$$

This is the equation of the projection on the plane XY of the locus of the intersection of consecutive characteristics. Its projection on the plane ZX may be found by eliminating m and y from the three equations, (351), (352), (354). Represent the result of this elimination by

(356) $$f(z,x) = o,$$

and we have the two projections (355), (356), which make known the locus required.

This locus is by some mathematicians called the *Edge of Regression* of the envelope, because it is the curve on the envelope to which all the characteristics are tangent, and consequently is the boundary of the envelope. Equations (355), (356), are the Edge of Regression of (353). The envelope (353) may, therefore, be regarded as made up of the tangents to the Edge of Regression. We will employ this consideration in a subsequent proposition.

It is obvious, that if a particular value be given to m in (351), (352), (354), the values of x,y,z determined from these three equations will make known a point on the Edge of Regression. The curve may in this way be determined by points.

PROPOSITION LXVII.

A plane touches two given curves; determine its envelope.

A curve situated in any manner in space, may be represented by its projections on two of the co-ordinate planes. Let XZ and YZ be the planes on which the projections are made. Let the projections of one curve be,

(357) $$y' = \phi z', \qquad x' = \psi z',$$

and of the other,

(358) $$y'' = \mathrm{F}z'', \qquad x'' = fz''.$$

Let the equation of the required plane be,

(359) $$z = mx + ny + \mathrm{B}.$$

We must first determine two of the parameters m,n,B, in terms of the third. For this purpose, observe that where the plane (359) touches the curve (357), the co-ordinates are common to the plane and curve, and (359) may be written,

(360) $$z' = mx' + ny' + \mathrm{B}.$$

In like manner, where (359) touches (358), we may write (359),

(361) $$z'' = mx'' + ny'' + \mathrm{B}.$$

Observe, also, that the tangent line to the curve (357), at the point of contact of (357) and (359), lies in the plane (359).

The projections of this tangent line are tangent to the projections (357) of the curve, and the equation of the tangent line in space will be the equations of the two tangent lines to the two projections (357). The equation of these tangent lines are, as at (65),

(362) $$y - y' = \frac{dy'}{dz'} \cdot (z - z'), \text{ and } x - x' = \frac{dx'}{dz'} \cdot (z - z').$$

These together represent the tangent line to the curve whose projections are (357).

And since the line (362) coincides with plane (359), we have, [from Analytical Geometry], by the condition of the coincidence of a line with a plane,

(363) $$m \frac{dx'}{dz'} + n \frac{dy'}{dz'} - 1 = o.$$

Similarly for the coincidence of the tangent to (358) with (359), we have,

$$(364) \qquad m \frac{dx''}{dz''} + n \frac{dy''}{dz''} - 1 = o.$$

Eliminate x', y', and z', from the four equations (357), (360), and (363), and the resulting equation will contain m, n, and B, and may be represented by

$$(365) \qquad \phi(m,n,\mathrm{B}) = o.$$

In like manner, from (358), (361), (364), we have,

$$(366) \qquad \psi(m,n,\mathrm{B}) = o.$$

By means of (365) and (366), two of the parameters may be eliminated from (359), and the envelope found, as in Prop. LXV.

As a particular example, let the curve (357) be,

$$(g) \qquad x' = az'^2, \qquad y = bz'^2,$$

and the curve (358),

$$(h) \qquad x'' = cz''^2, \qquad y'' = ez''^2.$$

The particular values of (365) and (366), determined as above directed are,

$$(k) \qquad 2\mathrm{B}\,(am + bn) - 1 = o, \qquad 2\mathrm{B}\,(cm + en) - 1 = o.$$

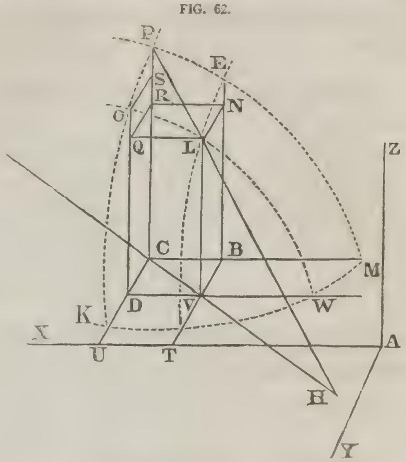
Solve equations (k) for n and B, and their values put into (359), we have the form of the tangent plane, in this particular case, involving the parameter m. This parameter being eliminated by the principle of Consecutive Surfaces, we have the envelope required.

PROPOSITION LXVIII.

Determine the increment of the ordinate in passing from one point to another, on a given curve surface.

This Proposition leads to the development of functions of two variables.

FIG. 62.



Let the equation of the surface PMK be

(366) $z = \varphi(x,y)$.

For the point L on the surface the co-ordinates are, $AT = x$, $TV = y$, and $VL = z$.

In passing from L to P on the surface, the ordinate VL is increased by PR. It is required to find the length of PR.

Through L and P pass planes parallel to the co-ordinate planes ZX and ZY.

If x receive the increment $h = TU$, while y remains constant, the z in (366) becomes OD, which is the ordinate in the parallel section. Put $OD = z'$, and by Taylor's Theorem, [see (218)], equation (366) becomes

$$(367) \quad z' = \varphi(x + h,y) = z + \frac{dz}{dx} h + \frac{d^2z}{2dx^2} h^2 + \&c.$$

If in (366) y receive an increment $k = VB$, while x remains constant then z becomes EB which is the ordinate in the other parallel section. Put $EB = z''$, and by Taylor's Theorem, (366) becomes

$$(368) \quad z'' = \varphi(x,y + k) = z + \frac{dz}{dy} k + \frac{d^2z}{2dy^2} k^2 + \&c.$$

If we transpose z in (367) and (368), we observe, at once the increment of the ordinate in each of the parallel sections.

From (367) or (368), we observe, that to pass from one point to another on a parallel section, the second ordinate (z' or z'') is equal to the first ordinate, plus the successive differential coefficients of the parallel section multiplied by the increment of the abscissa of the section. Hence, to pass from O to P, in the parallel section OP, put $PC = z'''$, and by analogy to (367), we have,

(369) $z''' = z' + \frac{dz'}{dy}k + \frac{d^2z'}{2dy^2}k^2 + \text{\&c.} = \varphi(x+h, y+k).$

To pass from E to P in the parallel section EP, we have in like manner,

(370) $z''' = z'' + \frac{dz''}{dx}h + \frac{d^2z''}{2dx^2}h^2 + \text{\&c.} = \varphi(x+h, y+k).$

Differentiate (367) for x constant and y variable, (y being supposed to enter into the value of z, d^2z, &c.) and we have,

(370a) $$\frac{dz'}{dy} = \frac{dz}{dy} + \frac{d^2z.h}{dx\,dy} + \frac{d^3z.h^2}{2dx^2dy} + \text{\&c.}$$

Differentiate (370a) again for x constant and y variable, and we have [y being the independent variable],

(370b) $$\frac{d^2z'}{dy^2} = \frac{d^2z}{dy^2} + \frac{d^3z.\,h}{dxdy^2} + \text{\&c.}$$

If we were to differentiate (370b) again, we would have the third differential coefficient in respect of y, &c.

For $$z',\ \frac{dz'}{dy},\ \ \frac{d^2z'}{dy^2},\ \text{\&c.},$$

in (369) put their values in (367), (370a), (370b), and we have,

(371) $$\begin{cases} z''' = \varphi(x+h, y+k) = z + \frac{dz}{dx}h + \frac{dz}{dy}k + \\ \frac{d^2z}{2dx^2}h^2 + \frac{d^2z\,hk}{dx\,dy} + \frac{d^2z}{2dy^2}k^2, \text{\&c.} \end{cases}$$

Were we to differentiate (368) for y constant and x variable, and then put into (370) for z'', $\frac{dz''}{dx''}$, $\frac{d^2z''}{dx^2}$, &c., their values from (368) and its differentials, we have,

(371a) $$\begin{cases} z'''=\varphi(x+h, y+k) = z + \frac{dz}{dx}h + \frac{dz}{dy}k + \frac{d^2z}{2dx^2}h^2 + \\ \frac{d^2z\,hk}{dy\,dx} + \frac{d^2z}{2dy^2}k^2 +, \text{\&c.} \end{cases}$$

The developments (371) and (371a) are equivalent, each being the length of the same ordinate PC or z'''.

Equation (371) or (371a), is the length of the ordinate PC, in

terms of the ordinate at the point L, and of the increments and differentials in the parallel sections through L.

If we put,

(372) $$\frac{dz}{dx} = p,\ \frac{dz}{dy} = q,\ \frac{d^2z}{dx^2} = r,\ \frac{d^2z}{dx\,dy} = s,\ \text{and}\ \frac{d^2z}{dy^2} = t,$$

and use only the first six terms of (371) or (371*a*), we have,

(373) $$z''' - z = ph + qk + \tfrac{1}{2}(rh^2 + 2shk + tk^2) + \&c., = \mathrm{PR},$$

which exhibits the increment of the ordinate in passing from one point to any other on a curve surface.

Ex.—Let the equation (366) be

(*a*) $$z = x^n\, y^m.$$

If x receive the increment h, and y the increment k, (*a*) becomes

(*b*) $$z''' = (x + h)^n\, (y + k)^m,$$

which may be developed by formula (371) or (371*a*).

Cor. If the developments (371) and (371*a*) be equated the coefficients of the like powers of h and k are equal. This gives the condition.

(373*a*) $$\frac{d^2z}{dx\,dy} = \frac{d^2z}{dy\,dx}.$$

But the first side of (371*b*) is the differential of $\frac{dz}{dy}$ for y constant, and x variable, the second is the differential of $\frac{dz}{dx}$ for y variable and x constant, and (371*b*) shows these differentials to be equal. This may be exhibited in a particular case, as as follows:

Differentiate (*a*) above for x variable, and we have,

(*c*) $$\frac{dz}{dx} = nx^{n-1}\, y^m.$$

Differentiate (*a*) for y variable, and we have,

(*d*) $$\frac{dz}{dy} = mx^n\, y^{m-1}.$$

Differentiate (*c*) for y variable and x constant, and we have,

(*e*) $$\frac{d^2z}{dx\,dy}\ mn\ x^{n-1}\, y^{m-1}.$$

Differentiate (*d*) for x variable, and y constant, and we have,

(*f*) $$\frac{d^2z}{dy\,dx} = nm\ x^{n-1}\, y^{m-1}.$$

Equations (e) and (f) show the values of these differential coefficients to be equal, as in (371b). This principle is of much importance in the Integral Calculus.

PROPOSITION LXIX.

Determine the differential equations of the parallel and inclined sections of a curve surface.

Definitions.

1st. If the equation of a curve be differentiated the result is called the differential equation of the curve. Thus, $2xdx + 2ydy = o$, is the differential equation of the circle $x^2 + y^2 = R^2$.

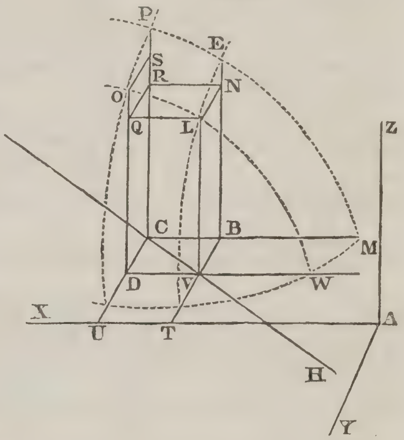


FIG. 64.

2d. We have defined (Proposition XLV.) a *parallel section* of a curve surface to be one whose plane is parallel to one of the co-ordinate planes ZX or ZY, and *a parallel trace* to be the trace on XY, of the plane of the parallel section. Let the section whose plane is perpendicular to the plane XY, and oblique to the other co-ordinate planes be called *an Inclined Section.*

Let the trace on XY, made by the plane of the inclined section, be called, *The Inclined Trace*, and let the plane which cuts the inclined section be called, *The Inclined Plane.* In the figure, if L and P be two points on the curve surface, LPCH is the inclined plane, CH is the inclined trace, and the section through L and P made by this plane, is the inclined section.

Let the equation of the curve surface be,

(374) $$z = \phi(x,y).$$

If (374) be differentiated for x constant, and y variable, we have,

(375) $$dz = \frac{d.\phi(x,y).dy}{dy} = \frac{dz}{dy}.\, dy = qdy.$$

If (374) be differentiated for y constant, we have,

(376) $$dz = \frac{d.\phi(x,y)\ .dx}{dx} = \frac{dz}{dx}\, dx = pdx.$$

Equation (375) is the differential equation of the parallel section EL, and (376) of LO.

If the points O and E be indefinitely near to L, then VD is the representative of dx in (376), VB of dy in (375), OQ of dz in (376), and EN of dz in (375).

For the differential equation of any section through L and P, differentiate (374) for x and y, both variable, and the total differential being the sum of the partial differentials, this differential is,

(377) $$dz = pdx + qdy.$$

Here dz would be represented by PR.

Equation (377) is the differential equation of a section made by a plane cutting the surface in any manner, in other words, it is the differential of the surface itself, and is called the total differential of (374).

To obtain the differential equation of an inclined section, let the equation of the inclined plane PCH be,

(378) $$y = \mathrm{A}x + \mathrm{B},$$

which is the equation of a plane perpendicular to plane XY.

From (378), we have, by differentiating

(379) $$dy = \mathrm{A}dx.$$

This value substituted into (377), we have,

(380) $$dz = (p + \mathrm{A}q)dx,$$

which is the differential equation of the inclined section made by plane (378).

Ex.—Determine the differential equation of the inclined section made by plane (378) with the following surfaces :

(Q) $$\left\{\begin{array}{l} x^2 + y^2 - 4mz = o, \qquad x^2 + y^2 - m^2 z^2 = o, \\ x^2 + y^2 + z^2 - \mathrm{R}^2 = o, \\ a^2 z^2 + b^2 x^2 + b^2 z^2 - a^2 b^2 = o. \end{array}\right.$$

From the first of (Q,) we have,

$$\frac{dz}{dx} = \frac{x}{2m} = p, \quad \text{and} \quad \frac{dz}{dy} = \frac{y}{2m} = q.$$

These values substituted into (380), we have,

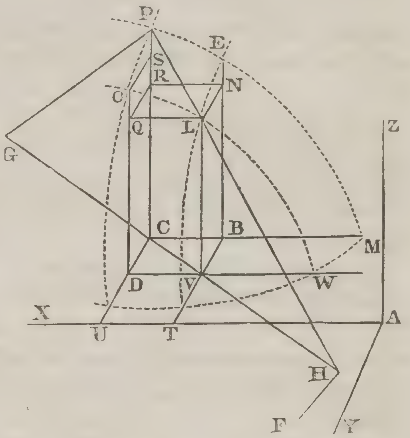
$$dz = (x + \mathrm{A}y)\frac{dx}{2m},$$

for the differential equation of the section.

PROPOSITION LXX.

Determine the length of the subtangent at a given point on the inclined section of a curve surface.

FIG. 65.



Let P be the given point, and

(381) $z = \phi(x,y)$

the equation of the surface. Let (378) be the inclined plane cutting the section, and (380) the differential equation of the section. Let PH be the tangent line at P to the inclined section, then CH is the subtangent whose length is required.

If we suppose the inclined trace CH to be the axis of abscissas of the inclined section, and call it the axis of S, then CV $= d$S. If the co-ordinate planes be rectangular, CDV is a right-angled triangle, and since VD $= dx$, and DC $= dy$, we have,

(382) $$d\mathrm{S}^2 = dx^2 + dy^2.$$

Substitute (379) into (382), and we have,

(383) $$d\mathrm{S} = dx\,(1 + \mathrm{A}^2)^{\frac{1}{2}}$$

From similar triangles PRL and PCH, we have,

(384) PR : RL :: PC : CH, or $dz : d\mathrm{S} :: z :$ subtangent.

By means of (383) and (380), we have, from (384),

(385) $$\text{subtang.} = \frac{z(1 + A^2)^{\frac{1}{2}}}{p + qA} = CH.$$

This is the length of the subtangent required, involving the partial differential coefficients p and q, which are to be deduced from the equation of the surface (381).

Ex.—Determine the subtangent in the inclined section of surfaces (Q), last Proposition.

PROPOSITION LXXI.

Determine the length of the subnormal at a given point on the inclined section of a curve surface.

From the point P, fig. 65, draw PG perpendicular to PH, and in the plane of the inclined section PHC. Then CG is the subnormal whose length is required.

Since the triangle GPH is right-angled at P, we have,

(386) $$GC = \frac{\overline{PC}^2}{CH}.$$

But PC is the ordinate z. Substitute into (386), from (385), the value of CH, and we have,

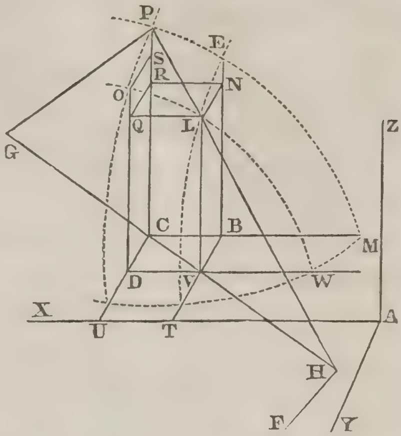
(387) $$\text{subnormal} = \frac{z(p + qA)}{(1 + A^2)^{\frac{1}{2}}}.$$

Ex.—Determine the subnormal in the inclined section of surfaces (Q).

PROPOSITION LXXII.

Determine the position of the inclined plane through a given point on a curve surface, when the subtangent of the inclined section is a minimum.

FIG. 66.



Putting T for CH, the length of the subtangent, we have from (385),

$$(388) \quad \mathrm{T} = \frac{z\,(1+\mathrm{A}^2)^{\frac{1}{2}}}{p + q\mathrm{A}}.$$

For the point P, the ordinate z is constant, and p and q the differential coefficients of the parallel sections are also constant.

Hence for the point P, the subtangent (388) is a function of A, the tangent of the angle BCV.

Differentiate (388) for A variable, and putting its differential equal to zero, we find,

$$(389) \qquad \mathrm{A} = q \div p.$$

This value of A, substituted into (388), we have,

$$(390) \qquad \mathrm{T} = \frac{z}{(p^2 + q^2)^{\frac{1}{2}}}.$$

The value of A in (389), gives the position of the inclined plane (378), when the subtangent is a minimum, as may be verified by substituting, according to Proposition XXXI., (389) into the second differential of (388).

The value of A in (389) changes (378) to

$$(391) \qquad y = \frac{q}{p}\,x + \mathrm{B}.$$

This is the plane of the inclined section whose subtangent CH, is a minimum.

The plane of an inclined section through P perpendicular to (391), is

$$(392) \qquad y = -\frac{p}{q}\,x + \mathrm{C}.$$

But the value,

$$A = -p \div q,$$

substituted into (388), renders the subtangent infinite.

Hence the tangent line at P, to the inclined section made by plane (392), is parallel to the plane XY.

Let us call the inclined section made by plane (391), *The Minimum Section*, and plane (391) *The Minimum Plane.*

Corrollaries.

1.—The tangent line at P to the minimum section, is shorter than the tangent line to any other inclined section through that point.

For in the right-angled triangle PCH, if PC remain constant, PH is a minimum when CH is.

2.—The tangent line at P to the minimum section has a greater inclination to the plane XY than any other tangent through that point.

For in the right-angled triangle PCH, if PC remain constant, the angle PHC is greatest when CH is least. But the angle PHC is the inclination of the tangent line to the plane XY.

3.—The inclined sections that have the greatest and least tangents are perpendicular to each other.

4.—If a tangent plane to the surface be drawn through any point of the minimum section, its trace HF on the plane XY will be perpendicular to the tangent line to the minimum section. For since all the tangents to the surface at the point P lie in the same plane, and the tangent of the minimum section is the shortest, it is perpendicular to the trace HF.

The minimum section is the curve on which a heavy body on a curve surface moves when the plane XY is horizontal. This section is often used in Mechanics.

By substituting into (380) the value of A in (389), we find for the differential equation of the minimum section,

(393) $$dz = (p^2 + q^2)\,\frac{dx}{p}.$$

By equating the values of A in (389) and (379), we have,

(394) $$p\,dy + q\,dx = o,$$

which may also be used as the differential equation of the same section.

Ex.—Determine the length of the minimum subtangent in surfaces (Q).

PROPOSITION LXXIII.

Determine the position of the inclined plane through a given point on a curve surface, when the subnormal of the inclined section is a maximum.

This is determined by differentiating (387) for A variable, and equating the differential to zero. The value of A is found to be,

(395) $$A = q \div p,$$

which renders (387) a maximum.

Comparing (395) and (389), we observe, that the inclined section which has a maximum subnormal is the one which has a minimum subtangent—a result which might have been observed immediately from the diagram fig. 66 ; for in the right angled triangle PGH, if PC remain constant, CG is greatest when CH is least. The same truth is manifest in (386).

PROPOSITION LXXIV.

Determine the second differential of an inclined section of a curve surface.

Equation (377), viz.

(396) $$z = pdx + qdy,$$

may be taken as the first differential of any section of a curve surface. It is usually regarded as the differential of the surface, being that of a section of the surface made by a plane cutting it in any manner. As x and y are the independent variables, dx and dy are constant. Differentiate (396) for dx and dy constant, and we have,

(397) $$\therefore d^2z = dp\, dx + dq\, dy.$$

(397a) $$\text{But } p = \frac{dz}{dx}, \text{ and } q = \frac{dz}{dy}.$$

The differential of p is the sum of its partial differentials, first for

x, and second for y variable. Hence, we have, from the first of (397a),

(398) $$\therefore\ dp = \frac{d^2z}{dx^2} . dx + \frac{d^2z}{dx\,dy} . dy.$$

Using the notation at (372) for the partial second differential coefficients (398) becomes

(399) $$dp = r\,dx + s\,dy.$$

By differentiating $q = \frac{dz}{dy}$ in the same manner and using notation (372), we have,

(400) $$dq = s\,dx + t\,dy.$$

Substitute (399) and (400) into (397), and we have,

(401) $$d^2z = r\,dx^2 + 2s\,dx\,dy + t\,dy^2.$$

This is the second differential of any section.

It must be carefully observed that r, s and t in these equations are the differential coefficients of the parallel sections, and that d^2z on the first side of (401,) and (397) is the second differential of z in any section.

By proceeding with (401) as we have here done with (396), we should obtain the third differential of any section, &c.

To obtain the second differential of the inclined section made by plane (378), substitute into (401) for dy, its value in (379), and we have,

(402) $$d^2z = r\,dx^2 + 2s\,\mathrm{A}dx^2 + t\,\mathrm{A}^2dx^2.$$

This is the second differential required.

Ex.—Determine the second differential of the inclined section made by plane (378) with the surfaces,

$$x^2 + y^2 - 4mz = o, \qquad x^2 + y^2 - m^2z^2 = o.$$

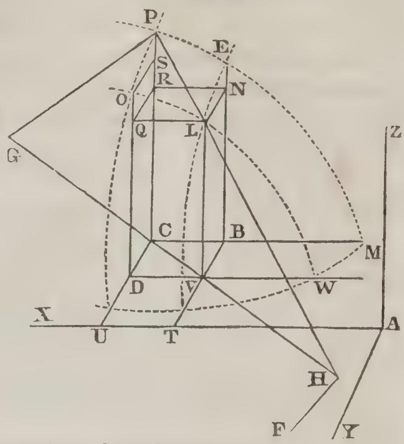
PROPOSITION LXXV.

Determine the radius of curvature at a given point on an inclined section of a curve surface.

If β be the ordinate, and α the abscissa of any point on a plane curve, the radius of curvature is, [see (199),]

(403) $$R = \pm \frac{(da^2 + d\beta^2)^{\frac{3}{2}}}{da\, d^2\beta}.$$

FIG. 67.



If the inclined trace CH be taken as axis of abscissas, and called the axis of S, and the ordinate of any point P on the inclined section be z, the a and β of (403) become S and z in the inclined section, and (403) becomes,

(404) $$R = \pm \frac{(dS^2 + dz^2)^{\frac{3}{2}}}{dS\, d^2z}.$$

Substitute into (404) for dS, dz, and d^2z, their values in (383), (380), and (402), and we have,

(405) $$R = \pm \frac{\left(1 + A^2 + (p + qA)^2\right)^{\frac{3}{2}}}{\left(r + 2sA + tA^2\right).\left(1 + A^2\right)^{\frac{1}{2}}}.$$

This is the radius of curvature required.

If we substitute into (405), for A, its value in (389), we have, for the radius of curvature of the minimum section,

(406) $$R = \pm \frac{(1 + p^2 + q^2)^{\frac{3}{2}}(p^2 + q^2)}{r\, p^2 + 2s\; pq + t\, q^2}.$$

Ex.—Determine the radius of curvature in the minimum sections of surfaces,

$$x^2 + y^2 - 4mz = o, \qquad x^2 + y^2 - m^2 = o.$$

The first of these surfaces gives for the radius of curvature of the minimum section,

$$R = 2\,(m + z)^{\frac{3}{2}} \div m^{\frac{1}{2}}.$$

PROPOSITION LXXVI.

Determine the radius of curvature of a normal section through a given point on a curve surface.

Definition.—A normal section through a point on a curve surface is one whose plane contains the normal to the surface at the point. It is obvious there may be an indefinite number of sections through a point containing the same normal.

This is merely a particular case of the preceding Proposition, and the solution may be obtained from (405), as follows.

Suppose the planes of reference so taken that the normal line to the given surface, at the proposed point, be perpendicular to the plane XY. Then (378) may be taken as the plane of the normal section.

Suppose P, [fig. 67], be the proposed point, and suppose the ordinate PC be normal to the surface, then all the tangent lines to the surface at P will be parallel to the plane XY. Consequently for this point, we have,

(407) $$p = o, \qquad \text{and} \qquad q = o,$$

because p and q expresses the tangents of the angles which the tangent lines to the parallel sections make with the parallel traces, and these angles being zero, or 180°, we have (407).

The conditions (407) reduce (405) to

(408) $$R = \pm \frac{1 + A^2}{r + 2sA + tA^2}.$$

This is the radius of curvature required.

PROPOSITION LXXVII.

Determine the normal section whose radius of curvature is a maximum.

The second differential coefficients r, s, and t, appertaining to the parallel sections through any point P, [fig. 67], are constant for that point. Hence the radius of curvature in (408), is for the same point P, a function of A. Hence by the rule for a maximum, differentiate (408) for A variable, and putting the differential equal to zero, we have,

(409) $$A^2 + \frac{r - t}{s} . A - 1 = o.$$

This solved, gives two values of A, one of which gives a maximum, and the other a minimum radius of curvature. [See Proposition XXXI.]

By the theory of equations, the product of the roots of a quadratic equals the last term of the equation. Hence put A' and A'' for the roots of (409), and we have,

(410) $$A'A'' = -1,$$

which exhibits the well known relation of the tangents of perpendicular lines. Hence the two planes given by the two values of A in (409), are perpendicular to each other, and we have the remarkable property that, *The normal sections of greatest and least curvature are perpendicular to each other.*

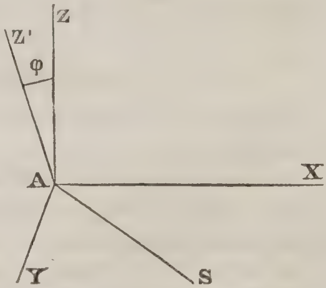
To obtain the length of the maximum and minimum radii of curvature, substitute into (408) the values of A obtained by solving (409).

PROPOSITION LXXVIII.

Determine the radius of curvature at a given point of an oblique section.

Definition.—An oblique section of a curve surface is one whose plane is inclined in any given manner to the co-ordinate planes.

FIG. 68.



For convenience, suppose the point at which the radius of curvature is drawn to be at the origin, and the plane of XY to be tangent to the surface at that point. Then the axis of Z will coincide with the normal.

Suppose the plane of the oblique section make an angle φ with the axis of Z, and let its trace AS on XY, be the axis of abscissas S, of the oblique section, and AZ′, the axis of ordinates of the same section.

The radius of curvature in the oblique section is as in (404), which may be put in the form,

(411) $$R = \pm \frac{ds^2}{d^2z'} \cdot \left(1 + \frac{dz'^2}{ds^2}\right)^{\frac{3}{2}}.$$

But since $\frac{dz'}{ds}$ expresses the tangent of the angle which the tangent line makes with the axis of abscissas, which angle in this case, is zero, (411) becomes,

(412) $$R = \pm \frac{ds^2}{d^2z'}.$$

For the radius of curvature R′ of the normal section at A, whose plane has the same trace AS, we have in like manner,

(413) $$R' = \pm \frac{ds^2}{d^2z}.$$

But if a given distance on the axis of z' be projected orthogonally on the axis of z, we have,

$$z = z' \cos.\varphi.$$

Differentiating this twice, we have,

$$d^2z = d^2z' \cos.\varphi.$$

Substitute this value of d^2z' into (412), and we have,

(414) $$R = \pm \frac{ds^2}{d^2z} \cos.\varphi.$$

Comparing (413) and (414), we observe that the radius of curvature of the oblique section is the projection of the radius of curvature of the normal section, on the plane of the oblique section. This supposes the trace of the planes of the normal and oblique sections on the plane of XY, to be the same.

PROPOSITION LXXIX.

Determine the direction of a line of curvature at a given point on a curve surface.

Definitions.

1st. Let us call the point on a curve surface at which a normal line is drawn, *The Normal Point.*

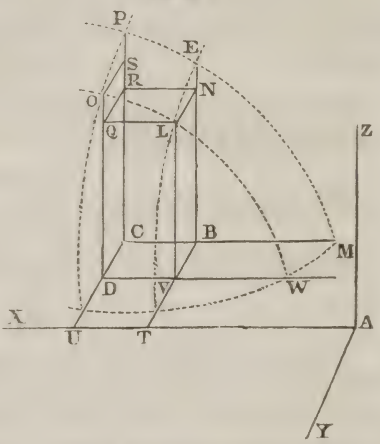
2d. A line of curvature is the locus of the normal points of consecutive intersecting normals.

This definition is necessary for the reason that consecutive normals to a surface do not necessarily intersect. They may be in different planes and then no intersection occurs. If they do intersect, their normal points will be on a curve (viz. the line of curvature), traced on the surface.

Suppose P be the given point. The equations of normal line through P are,

$$(415)\qquad \begin{cases} x' - x + p\,(z' - z) = o, \\ y' - y + q\,(z' - z) = o, \end{cases}$$

FIG. 69.



in which x', y' and z' are the variable co-ordinates of the normal line, and x, y, and z, the co-ordinates of the normal point P.

For the consecutive normal the x, y, z, p and q of (415), change when the normal point is changed and x', y', and z', are common to (415) and its consecutive normal at the point of intersection.—Hence, differentiate (415) for x', y' and z' constant, and we have,

$$(416)\qquad \begin{cases} -dx - pdz + (z' - z)\,dp = o, \\ -dy - qdz + (z' - z)\,dq = o. \end{cases}$$

By the principle of consecutive lines, x', y', and z', are common to (416) and (415) at the point of intersection of (415) with its consecutive normal.

Eliminate x', y', and z' between the four equations (415), (416) and the result will be the relation among the co-ordinates and differential coefficients of the normal point when the consecutive normals intersect.

As x' and y' do not enter into (416), it is sufficient merely to eliminate z' from the two equations (416), and we have,

(417) $$dp\,(dy + qdz) - dq\,(dx + pdz) = o.$$

If we substitute into (417) the values of dp and dq in (399) and (400), and the value of dz in (396), we will have an equation containing x, y and $\frac{dy}{dx}$, which may be represented by

(418) $$\phi\left(x,\, y,\, \frac{dy}{dx}\right) = o.$$

If (418) contain z, let the surface be

(419) $$z = \psi(x,y),$$

and z can be eliminated from (418).

Equation (418) is the differential equation of the projection of the line of curvature on the plane XY. If (418) be divisible into two factors, representing these factors by

(420) $$\mathrm{F}\left(x,y,\, \frac{dy}{dx}\right) = o, \quad \text{and } f\left(x,y,\, \frac{dy}{dx}\right) = o,$$

these two equations would indicate two lines of curvature on surface (419) at the same point.

If we suppose the co-ordinate planes so taken that the normal line on the proposed surface may be perpendicular to the plane of XY, then $p = o$, and $q = o$, and (417) becomes after substituting into it from (399) and (400),

(421) $$\frac{dy^2}{dx^2} + \frac{r - t}{s} \cdot \frac{dy}{dx} - 1 = o.$$

Here $\frac{dy}{dx}$ expresses the tangent of the angle made with the axis of X by the tangent line to the projection on XY of the line of curvature. This is the A of (378). And as in (409), so here (421) denotes that there are two tangent lines perpendicular to each other. Consequently, at the same normal point there are two lines of curvature at right angles to each other.

From the coincidence of (421) and (409), we infer, that the lines of curvature at any point, are tangents respectively to the sections of greatest and least curvature at the same point; and that the intersections of consecutive normals occur at the centres of

curvature of the sections of greatest and least curvature. This property enables us to determine the absolute length of the maximum and minimum radii of curvature at any point on a curve surface without changing the position of the co-ordinate axes, as was done to obtain (408) and (409). For, if x', y' and z' be the co-ordinates of the centre of curvature, we have, for the radius of curvature,

$$(422) \qquad R^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2.$$

Eliminate $\frac{dy}{dx}$ between the two equations (416), and then by means of this resulting equation, and (415), eliminate x', y', and z' from (422). The result will be the maximum and minimum radii of curvature.

As a particular example of the determination of lines of curvature, take the paraboloid of revolution whose axis of revolution is the axis of Z. Its equation is,

$$(a) \qquad x^2 + y^2 = 2mz.$$

Derive from (a) the differentials of (417), and we have,

$$(b) \qquad \frac{dz}{dx} = \frac{x}{m} = p \quad \therefore \quad dp = \frac{dx}{m},$$

$$(c) \qquad \frac{dz}{dy} = \frac{y}{m} = q \quad \therefore \quad dq = \frac{dy}{m},$$

$$(d) \qquad dz = \frac{x}{m}\,dx + \frac{y}{m}\,dy.$$

Substitute these differentials into (417), and we have an equation which may be divided into two factors, and which furnishes the equations,

$$(e) \qquad \frac{dy}{dx} = \frac{y}{x},$$

$$(f) \qquad \frac{dy}{dx} = -\frac{x}{y}.$$

Equations (e) and (f) are the particular forms of (420).

Equations (e) and (f) are properly the differentials of the projections on XY of the lines of curvature.

Equation (e) is the differential of a straight line through the origin. For every such line is of the form,

$$(g) \qquad y = cx \quad \therefore \quad \frac{dy}{dx} = c = \frac{y}{x}.$$

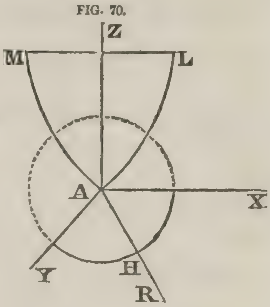
Equation (f) is obviously the differential of a circle, viz. of

$$(h) \qquad x^2 + y^2 = R^2.$$

Hence the projections on XY, of the lines of curvature on a paraboloid, are a straight line through the origin, and a circle with its centre at the origin.

This result we ought obviously to have. For the projection on XY, of the meridional section AL, of the paraboloid, is a straight line AR passing through the origin. And the projection on XY of a section of the paraboloid perpendicular to the axis of revolution AZ, is a circle having its centre at the origin.

FIG. 70.



And we know that normals to the meridional section AL, would be normals to the surface, and intersect in the plane of the meridional section, and that normals to the surface on a circle perpendicular to the axis of revolution, would intersect on the axis of revolution. The meridional section and circle perpendicular to it on the surface are therefore the lines of curvature of the paraboloid of revolution.

The same result would be obtained for any other surface of revolution except the sphere. For the sphere, all great circles are lines of curvature—a result that the analysis above would also make known.

PROPOSITION LXXX.

Determine a method for eliminating an indeterminate function.

Suppose we have the equations,

$$(423) \qquad u = fz, \quad \text{and}$$

$$(424) \qquad z = F(x,y),$$

in which x and y are independent variables, and fz an indetermi-

nate or unknown function of z. It is required to find a relation between u, x, and y, independent of the function fz.

The differential of (423) is,

(425) $$du = \frac{d\,fz\,dz}{dz},$$

and the two partial differentials of (424) are,

(426) $$dz = \frac{d\,\mathrm{F}\,(x,y).dx}{dx}, \qquad dz = \frac{d\,\mathrm{F}\,(x,y).dy}{dy}.$$

If for dz, in the numerator of (425), we put in succession the values of dz in (426), we have,

(427) $$\frac{du}{dx} = \frac{d\,fz}{dz}.\ \frac{d\,\mathrm{F}\,(x,y)}{dx}, \text{ and } \frac{du}{dy} = \frac{d\,fz}{dz}.\ \frac{d\,\mathrm{F}\,(x,y)}{dy}.$$

Divide equations (427), and we have,

(428) $$\frac{du}{dx} \div \frac{du}{dy} = \frac{d\,\mathrm{F}\,(x,y)}{dx} \div \frac{d\,\mathrm{F}\,(x,y)}{dy}.$$

This is the relation required, and is independent of the function fz.

Ex.—Let (423) be,

(*a*) $$u = \log.(a + z),$$

and let (424) be,

(*b*) $$z = mz + ny.$$

Then (428) becomes,

(*c*) $$\frac{du}{dx} \div \frac{du}{dy} = \frac{m}{n}.$$

This is an equation among the partial differential coefficients, and would be the same if, instead of (*a*), we had any other function of z.

Again, suppose we have the equations,

(*d*) $$\beta = \phi a, \qquad a = f\,(x,z), \qquad \beta = \mathrm{F}\,(y,z),$$

in which x and y are the independent variables.

If a relation be required among the variables x,y,z, in these equations, independently of φa, proceed as follows, [writing df for $df\,(x,z)$, and $d\,\mathrm{F}$ for $d\,\mathrm{F}\,(y,z)$.]

Differentiate equations (*d*) for y constant, and we have,

(*g*) $$d\beta = \frac{d\phi a.da}{da},$$

(h) $$da = \frac{df.dz}{dz}. + \frac{df.dx}{dx},$$

(k) $$d\beta = \frac{dF.dz}{dz}.$$

Again, differentiate equations (d) for x constant, and we have,

(l) $$d\beta = \frac{d\phi a.da}{da},$$

(m) $$da = \frac{df.dz}{dz}.$$

(n) $$d\beta = \frac{dF.dz}{dz}. + \frac{dF.dy}{dy}.$$

For da in the numerators of (g) and (l), substitute the value in (h) and (m) respectively, and we have from (g),

(o) $$\frac{d\beta}{dx} = \frac{d\phi a}{da}.\left(\frac{df}{dz}\,p + \frac{df}{dx}\right),$$

and from (l),

(r) $$\frac{d\beta}{dy} = \frac{d\phi a}{da} \cdot \frac{df}{dz}\,q.$$

From (k) we have,

(s) $$\frac{d\beta}{dx} = \frac{dF}{dz}.\,p,$$

and from (n),

(t) $$\frac{d\beta}{dy} = \frac{dF}{dz}.\,q + \frac{dF}{dy}.$$

Divide (o) by (r), and eliminate $\frac{d\beta}{dy}$ and $\frac{d\beta}{dx}$ by (s) and (t), and we have,

(429) $$\frac{dF}{dz}.\,p \div \left(\frac{dF}{dz}.\,q + \frac{dF}{dy}\right) = \left(\frac{df}{dz}\,p + \frac{df}{dx}\right) \div \frac{df}{dz}.\,q.$$

This is the relation among the partial differential coefficients, independently of ϕa.

An application of this general process will be given in the next Proposition.

PROPOSITION LXXXI.

Determine the equation of a cylindrical surface.

Definitions.

1st.—A cylindrical surface is one generated by a straight line moving parallel to itself along a given curve.

2d.—The curve along which the straight line moves is called, *The Directrix of the Cylinder.*

Let the curve in space, which is the directrix, be represented by the projections,

(430) $$x = \phi z, \qquad y = \psi z.$$

And let the equation of the generating line be,

(431) $$\begin{cases} x = mz + a, \\ y = nz + \beta, \end{cases} \quad \text{or,} \quad \begin{cases} a = x - mz, \\ \beta = y - nz. \end{cases}$$

Since the line is always parallel to itself, m and n are constant.

Where the line (431) meets the directrix (430), the co-ordinates are common to (430) and (431). Eliminate x, y, and z between these four equations, and we have a relation between a and β, which may be represented by

(432) $$\beta = \varphi a.$$

Substitute into this the values of a and β, in (431), and we have for the general equation of cylindrical surfaces,

(433) $$y - nz = \phi(x - mz).$$

If ϕa be eliminated between (431) and (432) by the principles of last Proposition, we have,

(433a) $$nq + mp - 1 = o,$$

for the relation among the partial differential coefficients of all cylindrical surfaces, whatever be the directrix. Equation (433a) is the differential equation of all cylindrical surfaces.

PROPOSITION LXXXII.

Determine the equation of a conical surface.

Let the equation of the directrix be

(434) $$x = \phi z, \quad \text{and } y = \psi z.$$

And let the equation of the line which generates the cone, be

$$(435) \qquad \begin{cases} x - x' = a(z - z'), \\ y - y' = \beta(z - z'), \end{cases}$$

where x', y', and z' are the vertex of the conical surface. Here a and β vary with the position of the generatrix. Eliminate x, y and z between (434) and (435), and the result may be written

$$(436) \qquad \beta = \phi a.$$

Substitute into this for β and a, their values in (435), and we have for the equation of conical surfaces.

$$(437) \qquad \frac{y - y}{z - z'} = \phi\cdot\left(\frac{x - x'}{z - z'}\right).$$

If ϕa be eliminated from (436) and (435) by (429), we have, after clearing of fractions,

$$(438) \qquad z - z' = p(x - x') + q(y - y').$$

This is the relation among the partial differential coefficients of all conical surfaces.

These two Propositions show the nature of the results obtained by eliminating an indeterminate function according to the process of Proposition LXXX. These results (in these two propositions) are deducible in a simple manner, independently of the Calculus. The first, (433a), is the condition of parallelism of a line to a plane, as it is obtained in Analytical Geometry. The last, (438), is the equation of a tangent plane through the vertex, and designates that the tangent plane to a conical surface coincides with the surface along the generatrix.

Many other Propositions are resolved by the process of elimination indicated in Proposition LXXX ; but the foregoing will suffice.

PROPOSITION LXXXIII.

Determine the equation of the tangent line to a curve of double curvature.

Let the equations of the projections of the curve of double curvature be,

$$(439) \qquad x = \phi z, \qquad y = \psi z,$$

The equations of the tangent lines to these projections are as in Proposition III.

$$(440)\qquad \begin{cases} z' - z = p\,(x' - x), \\ z' - z = q\,(y' - y). \end{cases}$$

These together represent the tangant line required, where $x'\ y'\ z'$, are the variable co-ordinates.

Ex.—Let the curve of double curvature be formed by the intersection of the surfaces,

$$(a)\qquad x^2 + z^2 = y^2,$$

$$(b)\qquad x^2 + y^2 + z^2 = 2\,Rz,$$

of which (a) is a cone whose vertical angle is 90°, and (b) is a sphere passing through the origin. The projections of their intersection are,

$$(c)\qquad x^2 + z^2 = Rz,$$

a circle, and

$$(d)\qquad y^2 = Rz,$$

a parabola.

The tangents to (c) and (d) are the projections of the tangents to the line of intersection of the cone and sphere.

PROPOSITION LXXXIV.

Determine the surface formed by the tangents to a curve of double curvature.

In Proposition LXVI, we observed that the surface formed by the characteristics of consecutive planes might be regarded as made up of the tangents to the Edge of Regression. The curve of double curvature may be regarded as the Edge of Regression of the surface formed by its tangents (440). Let the equations of the curve of double curvature be (439), and the equations of its tangent are,

$$(441)\qquad \begin{cases} x' - \varphi z = \varphi' z\,(z' - z), \\ y' - \psi z = \psi' z\,(z' - z), \end{cases}$$

$$\text{where } \varphi' z = \frac{d\varphi z}{dz} = p, \quad \text{and } \psi' z = \frac{d\psi z}{dz} = q.$$

The equations (441) contain the parameter z, which is the ordinate of the point of tangency. Eliminate z between the equations (441), and the result, which may be represented by

$$(442)\qquad \phi(x', y', z') = o,$$

is a relation among the co-ordinates of any point on the tangent

line, irrespective of the point of tangency. Consequently, (442) is the locus of all the tangents to the curve of double curvature.

Ex. 1.—Determine this locus when the curve is the intersection of the cone and sphere of last Proposition.

Ex. 2.—Determine this locus when the curve of double curvature is,

(c) $$x = az^2, \qquad y = bz^2.$$

From (c) we have, $p = 2az$, $q = 2bz$, and (441) becomes,

(d) $$x' - az^2 = 2az(z' - z), \qquad y' - bz^2 = 2bz\,(z' - z).$$

Multiply the first of (d) by b and the second by a and subtracting the results, we have,

(e) $$ay' = bx',$$

which is the locus required. Equation (e) being the equation of a plane perpendicular to the plane XY shows that the curve (c) is a plane curve. And in general the result of the elimination of z between equations (441) determines whether (439) is a plane curve or a curve of double curvature.

PROPOSITION LXXXV.

Determine the equation of the normal plane at a given point in a curve of double curvature.

Let the curve of double curvature be represented by (439). If at any point in a curve of double curvature a normal line be drawn, this normal may take an indefinite number of positions, all however, in the same plane. This plane is called the normal plane. If x' y' and z', be any point on the normal plane, its equation, is of the form

(443) $$z' - z = m\,(x' - x) + n\,(y' - y),$$

where x,y,z, are the normal point on the curve.

Let (440) be the tangent line to (439).

A plane perpendicular to a curve is perpendicular to the tangent to the curve at the normal point; and if a plane be perpendicular to a line, the traces of the plane are perpendicular to the projections of the line comparing the traces of (443) on ZX and ZY, with the projections of the tangent (440), we have,

$$n = -\frac{1}{q} = -\frac{dy}{dz}, \text{ and } m = -\frac{1}{p} = -\frac{dx}{dz}.$$

These values of m and n substituted into (443) give us for the equation of the normal plane,

$$(444) \qquad z' - z + \frac{dx}{dz}(x' - x) + \frac{dy}{dz}(y' - y) = o.$$

By substituting into (444), the values of x and y, and their differentials from (439), the equation of the normal plane to curve (439) is,

$$(445) \qquad z' - z + \varphi' z\,(x' - \varphi z) + \psi' z\,(y' - \psi z) = o.$$

Ex.—Let the curve of double curvature be,

$$(e) \qquad x = az^2, \qquad y = bz^2,$$

find the normal plane.

PROPOSITION LXXXVI.

Determine the surface generated by the intersection of consecutive normal planes to a curve of double curvature.

The normal plane (445), involves the parameter z, which is the ordinate of the normal point. Hence differentiate (445) for z variable. This differential, after dividing by dz, may be written,

$$(446) \qquad -1 - (\varphi' z)^2 - (\psi' z)^2 + \varphi'' z\,(x' - \varphi z) + \psi'' z\,(y' - \psi z) = o.$$

Eliminate the parameter z between (446) and (445), and we have,

$$(447) \qquad \varphi(x', y', z') = o,$$

for the surface required.

Equation (446) is the characteristic of plane (445).

PROPOSITION LXXXVII.

Determine the osculating plane at a given point in a curve of double curvature.

Definition.—A curve of double curvature may be considered as a polygon of an indefinite number of sides. At a given point of this curve, two consecutive sides, and no more, lie in the same plane. The plane which contains any two consecutive sides is called, *The Osculating Plane of the Curve of Double Curvature.*

Regard, as in Proposition XXXVII., the middle points of each of two consecutive sides of the polygon as consecutive points. A tangent line to the curve may be regarded as the prolongation of one of its indefinitely small sides.

Let x',y',z' be the given point on the curve of double curvature.

Let the equation of the curve be represented by

(448) $$x' = \varphi z', \qquad y' = \psi z'.$$

The equation of any plane through x',y',z' is of the form,

(449) $$z - z' = m\,(x - x') + n\,(y - y'),$$

where x, y, and z are any points on the plane.

The equation of the tangent line to (448) through x',y',z', is

(450) $$x - x' = \frac{dx'}{dz'}(z - z') \quad \text{and} \quad y - y' = \frac{dy'}{dz'}(z - z').$$

In order that the line (450) may coincide with plane (449), we have, from the principle of coincidence of a line and plane, [Analytical Geometry],

(451) $$m\,\frac{dx'}{dz'} + n\,\frac{dy'}{dz'} - 1 = o.$$

For the coincidence of plane (449) with the tangent line to (448), through any other point x'',y'',z'', we have in like manner,

(452) $$m\,\frac{dx''}{dz''} + n\,\frac{dy''}{dz''} - 1 = o.$$

When the points x',y',z', and x'',y'',z'' are consecutive, we have,

$$x'' = x' + dx', \qquad y'' = y' + dy', \quad \text{and} \quad z'' = z' + dz',$$

from which, regarding z' as the independent variable, we have, by differentiation,

(453) $$dx'' = dx' + d^2x', \quad dy'' = dy' + d^2y', \quad \text{and} \quad dz'' = dz'.$$

Substitute (453) into (452), and subtracting (451) from the result, we have,

(454) $$m\,\frac{d^2x'}{dz'^2} + n\,\frac{d^2y'}{dz'^2} = o.$$

The two relations (454) and (451) suffice to determine m and n, and their values put into (449) make known the osculating plane. If we put

(455) $$\frac{dx'}{dz'} = p, \quad \frac{dy'}{dz'} = q, \quad \frac{d^2x'}{dz'^2} = r, \quad \text{and} \quad \frac{d^2y'}{dz'^2} = t,$$

and solve (454) and (451) for m and n, we have,

(456) $$m = \frac{t}{pt - qr}, \quad \text{and} \quad n = -\frac{r}{pt - qr},$$

and hence (449) becomes,

(457) $$(pt - qr)(z - z') - t(x - x') + r(y - y') = o,$$

which is the osculating plane required.

Ex.—Determine the osculating plane to the curve,

$$x = az^2, \qquad y = (R^2 - z^2)^{\frac{1}{2}}.$$

PROPOSITION LXXXVIII.

Determine the radius of curvature at a given point on a curve of double curvature.

To a curve, considered in space, different osculating circles may be applied at the same point. The plane of the osculating circle, whose radius of curvature we here require, coincides with the osculating plane (457), determined in last Proposition. The radius of curvature must also lie in the normal plane to the curve of double curvature. Hence, it coincides with the intersection of the normal and osculating planes, and as the osculating circle must coincide with the curve at two consecutive points, (Prop. XXXVIII.), the centre of curvature must be on the characteristic or line of intersection of the normal plane with its consecutive plane.

Adopting the notation (455), the normal plane (444) may be written (x', y', z', being the normal point),

(458) $$z - z' + p(x - x') + q(y - y') = o,$$

and its differential corresponding to (446), is

(459) $$-(1 + p^2 + q^2) + r(x - x') + t(y - y') = o.$$

If x, y, and z, be the centre of the osculating circle, the radius of curvature calling it R, is

(460) $$R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2.$$

At the centre of curvature x, y, and z, are common to the osculating plane (457), to the normal plane (458,) to the characteristic (459), and to the radius of curvature (460). Hence, eliminate x, y,

and z, from (460) by means of (457), (458) and (459), and we have the radius of curvature required. The result of this elimination is,

(461) $$R^2 = \frac{(1 + p^2 + q^2)^3}{t^2 + r^2 + (pt - qz)^2}.$$

Ex.—Determine the radius of curvature of the curve,

$$x = az^2, \qquad y = bz + z^2.$$

PROPOSITION LXXXIX.

Determine the centre and radius of curvature of an osculating sphere.

Definition.

An osculating sphere to a curve of double curvature, is one whose surface contains three consecutive points of the curve.

Let the curve of double curvature be,

(462) $$y' = \varphi z', \quad \text{and } x' = \psi z'.$$

Let P, P', P'' be three consecutive points on (462), and let (a,β,c) be the centre of the sphere required. The equation of the sphere is of the form,

(463) $$R^2 = (a - x')^2 + (\beta - y')^2 + (c - z')^2.$$

Suppose the centre (a,β,c) be on the normal plane to (462). The equation of the normal plane (458) is, for the centre of the sphere,

(464) $$c - z' + p\,(a - x') + q\,(\beta - y') = o.$$

The characteristic (459) of this normal plane, is

(465) $$-(1 + p^2 + q^2) + r\,(a - x') + t\,(\beta - y') = o.$$

Differentiate (465), and we have for the intersection of (465), with its consecutive characteristic,

(466) $$(a - x')\,r' + (\beta - y')\,t' - 3\,(pr + qt) = o,$$

where $$r' = \frac{dr}{dz'}, \quad \text{and} \quad t' = \frac{dt}{dz'}.$$

Now, if the sphere whose centre is at the intersection of (466) of two consecutive characteristics, passes through one point P on (462), it also passes through the three consecutive points P, P', P'' on (462): for the two consecutive characteristics are on three consecu-

tive normal planes, which pass through P, P′, P″. The sphere whose centre is at the point (466), is therefore, the sphere required. The four equations (463), (464), (465) and (466), make known the values of a,β,c, and R, which determine the centre, and radius of the osculating sphere.

Cor. The locus of the centre of the osculating sphere is the Edge of Regression of the envelope of the normal plane. For the equations (464), (465), (466), are those employed in determining the envelope and Edge of Regression. [See Propositions LXXXVI, &c.]

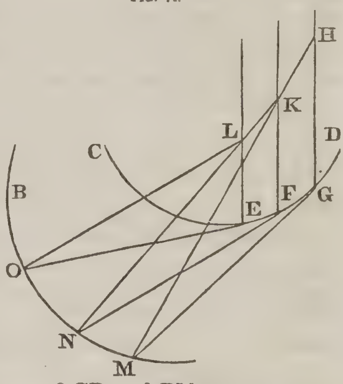
PROPOSITION XC.

Determine the evolute of a curve of double curvature.

If a cord be wrapped round the evolute of a plane curve, and continued to a point in the involute, the extremity of this cord will, in unwinding, describe the involute, [Proposition XL.]

The involute of a plane curve may likewise be described by the extremity of a cord unwound from a right cylinder, whose directrix is the evolute.

FIG. 71.



For let BM be the involute whose evolute in the same plane is CD. Through the consecutive points E,F,G, &c. erect lines EL, FK, GH, &c. perpendicular to the plane CD. These perpendiculars will form the envelope, or cylindrical surface that would be produced by the characteristics of normal planes to the curve BM. Let HGM be one of these normal planes, HG its characteristic, and GM its trace on the plane of CD and BM.

Suppose from any point H on GH, a line HM be drawn in the normal plane HGM to the involute at M. If we now suppose the normal plane HGM to be wrapped round the envelope HGEL, the

line GM will be wrapped round the evolute CD, and its extremity M will describe the involute MB; the line HM will be wrapped round the envelope HGEL, and its extremity M will likewise describe the involute MB. The line HM in this process will form on the envelope a curve HKL, which may also be regarded as the evolute of the curve MB. Hence any involute MB has an indefinite number of evolutes HKL, all traced on the envelope HGEL.

It is obvious that these evolutes will become straight lines when the envelope HGEL is developed or rolled out on a plane. Any line MH is tangent to the evolute HKL, for MH is the prolongation of HK, one of the indefinitely small sides of the polygon that composes the evolute.

If the curve BM be a curve of double curvature, and HGEL be the envelope of the normal plane to BM, take H any point on this envelope, draw HM in the normal plane to M, and suppose the normal plane HGM to be wrapped round the envelope, the point M will, as in the plane curve, describe the curve of double curvature MB. The curve HKL, traced on the envelope by the wrapping of the cord HM, will be the evolute of the curve MB. The evolute HKL will, as in the plane curve, have HM for its tangent, and become a straight line when the envelope HGEL is developed. There may obviously be an indefinite number of evolutes to BM, the curve of double curvature, all traced on the envelope HGEL.

To find the equation of any one of these evolutes, let x',y',z' be any point M on the given curve of double curvature, and x,y,z any point H on the evolute required. Let the curve MB be represented by

(467) $$y' = \phi z', \qquad x' = \psi z'.$$

The equation of the normal plane HGM is (458).

(468) $$z - z' + p\,(x - x') + q\,(y - y').$$

Since MH is tangent to the evolute HL, the equation of MH is for the point M,

(469) $$x' - x = \frac{dx}{dz}\,(z' - z),$$

(470) $$y' - y = \frac{dy}{dz}\,(z' - z).$$

Let the equation of the envelope HGEL, as found in Proposition LXXXVI. be,

(471) $$\phi(x,y,z) = o.$$

For the point M, the co-ordinates x', y', and z' are common to (467), (468), and (469); and for the point H we have x, y, and z common to (468), (469), and (471). Hence eliminate x',y',z' from (467), (468), (469). The result may be represented by

(472) $$\pi\left(x,y,z.\ \frac{dx}{dz}\right) = o.$$

If y be eliminated between (472) and (471), the result, which may be written,

(473) $$\psi\left(x,z.\ \frac{dx}{dz}\right) = o,$$

will be the differential equation of the projection of the evolute on the plane of XZ.

By the aid of the Integral Calculus, we obtain from (473), the projection of the evolute on XZ, which with surface (471), determines the evolute in space.

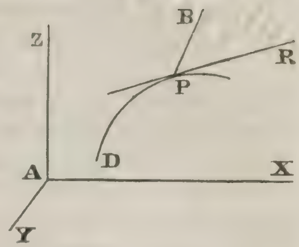
If G be a point where the osculating plane (457) to the curve at M, cuts the envelope HGEL, then is MG shorter than any other radius of curvature HM to the curve at M, i. e. MG is perpendicular to GH. This shortest radius is known by the name of *The Absolute Radius of Curvature*, and is the one whose length is determined at (461).

One property of these absolute radii of curvature is, that no two of them are in the same plane. Therefore, they do not intersect, consequently, the locus of the centres of the absolute radii of curvature is *not* an evolute of the curve.

PROPOSITION XCI.

A tangent is drawn to a curve of double curvature; at the point of tangency a line is drawn, making a given angle with the tangent, find the locus of the intersection of this line with its consecutive line.

FIG. 72.



Let the curve of double curvature be represented by

(474) $y = \varphi z, \quad x = \psi z.$

Let its projection on zx be PD.

Let PR be the projection of the tangent line.

Let PB be the projection of the line.

The equation of the tangent line is,

(475) $$Y - y = \frac{dy}{dz}\left(Z - z\right), \quad X - x = \frac{dx}{dz}\left(Z - z\right),$$

where x, y, and z are the co-ordinates of the point of tangency and the equation of the line making the given angle with the tangent, is (putting x', y', z' for its co-ordinates),

(476) $y' - y = b\,(z' - z), \qquad x' - x = a\,(z' - z).$

Put C for the cosine of the angle whose projection is BPR and [by Analytical Geometry,] we have for the angle included between the lines (475) and (476).

(477) $$C = \frac{1 + a\dfrac{dx}{dz} + b\,\dfrac{dy}{dz},}{(1 + a^2 + b^2)^{\frac{1}{2}}\left(1 + \dfrac{dx^2}{dz^2} + \dfrac{dy^2}{dz^2}\right)^{\frac{1}{2}}}.$$

By means of (474) the differentials, $\frac{dx}{dz}$, $\frac{dy}{dz}$, and x and y may be eliminated from (477), and the result solved for z may be represented by

(478) $$z = \phi(a, b),$$

which, of course, includes C and other constants.

Differentiate (476) for x', y', z' constant, and a and b variable, and eliminating $z' - z$ between the two differentials, we have,

(479) $$db\,\frac{dx}{dz} - da\,\frac{dy}{dz} = adb - bda,$$

which is the relation among the differentials when the line (476) intersects its consecutive line

By means of (474) we can eliminate,

$$\frac{dx}{dz}, \quad \frac{dy}{dz},$$

from (479), and then by means of (478) we can eliminate z from the result, and (479) may then be written,

(480) $$db\ \phi'(a,b) - da\ \psi'(a,b) = adb - bda.$$

Here again we must have recourse to the Integral Calculus, in order to obtain the equation whose differential is (480).

This original equation can contain no other variables than a and b. If we suppose this original equation to be obtained, and to be represented by

(481) $$b = fa,$$

we may solve (478) and (481) for a and b.

The result may be represented by

(482) $$b = \mathrm{F}z, \quad \text{and } a = fz.$$

By means of (482) and (474), the equation (476) of the line becomes

(483) $$y' - \phi z = \mathrm{F}z\,(z' - z), \quad \text{and} \quad x' - \psi z = fz\,(z' - z).$$

Differentiate either of the equations (483) for z variable, and eliminating z between this differential and each of the equations (483), we will have the projections of the curve required, which may be represented by

(484) $$y' = \mathrm{F}z', \qquad \text{and } x' = fz'.$$

Cor. If z be eliminated between the two equations (483), the result which may be represented by

(485) $$\mathrm{F}.(x', y'. z') = o,$$

is the developable surface which is the locus of the line (483), and of which (484) is the Edge of Regression.

PROPOSITION XCII.

A tangent line is drawn to a curve of double curvature, a right cone of given vertical angle has its vertex at the point of tangency, and the tangent line for its axis, determine the locus of the intersection of this cone with its consecutive cone.

Let the line (476) be the generatrix of the cone, and C being the cosine of half the vertical angle, we have as before, (477). If we substitute into (477) for a and b, their values from (476), the result is the equation of the cone, containing the three co-ordinates of the vertex. By means of (474), two of these co-ordinates, as x and y, may be eliminated, and we have the equation of the cone, which may be represented by

(486) $$\phi(x',y',z',\ z) = o.$$

Eliminate the parameter z between (486) and its differential for z, and we have the surface required, which may be represented by

(487) $$\mathrm{F}\ (x',y',z') = o.$$

Cor. 1.—Surface (487) contains all the lines (484).

Cor. 2.—If half the vertical angle is a right angle, then $\mathrm{C} = o$, and (477) becomes,

(488) $$1 + a\frac{dx}{dz} + b\,\frac{dy}{dz} = o,$$

and the cone becomes the normal plane, and the Proposition becomes Proposition LXXXVI., which is a particular case of the present.

PROPOSITION XCIII.

Determine the equation of a twisted surface.

A twisted surface is one generated by a straight line moving in a given manner, and continually changing the plane of its motion.

Let the equation of the generating line be,

(489) $$\begin{cases} x = az + b, \\ y = cz + e. \end{cases}$$

The four parameters a,b,c,e, which enter into (489), evidently change when the generatrix changes its position. If three of these parameters, as b,c,e, can be expressed in terms of the fourth, a, the parameter a can be eliminated between the two equations (489), and the result will be the surface required.

To express b, c, and e as functions of a, requires that the line (489) be subjected to three conditions.

The three conditions to which the generatrix (489) may be subjected, are either that it shall touch three given lines, or that it shall

touch two given lines, and be parallel to a given plane. This divides the Proposition into two cases.

Case 1st.

Let the generatrix (489) touch three given lines. Let the three lines be denoted by

(490) $$y = fz, \qquad x = \mathrm{F}z,$$

(491) $$y = f'z, \qquad x = \mathrm{F}'z,$$

(492) $$y = f''z, \qquad x = \mathrm{F}''z.$$

Where the line (489) touches (490), the x, y, and z are common to the four equations (489) and (490). Eliminate the three co-ordinates x, y, and z between these four equations. We may represent the result by

(493) $$\varphi(a,b,c,e) = o.$$

In like manner, and for a like reason, eliminate the three co-ordinates between the four equations (489) and (491), and we may represent the result by

(494) $$\varphi'(a,b,c,e) = o.$$

Again, eliminate the three co-ordinates between (489) and (492), and represent the result by

(495) $$\varphi''(a,b,c,e) = o.$$

Solve (493), (494), and (495), for b, c, and e, and represent the results by

(496) $$b = \varphi a, \qquad c = \psi a, \qquad e = \pi a.$$

Substitute the values of b, c, and e, from (496) into the generatrix (489), and we have,

(497) $$\begin{cases} x = az + \varphi a, \\ y = \psi az + \pi a. \end{cases}$$

Eliminate a between the two equations (497), and we have,

(498) $$\phi(x,y,z) = o,$$

the surface required.

The lines (490), (491), (492), may be either straight lines, or curved lines, or some of them straight and others curved.

Ex.—Let the three lines on which the generatrix moves be the three straight lines,

$$x = o, \qquad y = o,$$
$$x = mz, \qquad y = nz + q,$$
$$x = m'z + p, \qquad y = n'z.$$

Determine the generatrix (497).

Case 2d.

Let the generatrix (489) touch two given lines, and be parallel to a given plane.

Let (490) and (491) be the two given lines, and let the given plane be

(499) $$z = mx + ny + \mathrm{B}.$$

Since the generatrix (489) is to be parallel to plane (499), we have, by the conditions of this parallelism, [Analytical Geometry],

(500) $$ma + nc - 1 = o.$$

Derive (493) and (494) as before. Solve the three equations (493), (494), and (500), for b,c,e. Represent the results by (496), and we have as before, (497) for the directrix involving one parameter, and (498) for the surface.

Ex.—Let the two lines on which the generatrix moves be one the axis of y, and the other,

(A) $$x = mz + n, \quad \text{and} \quad y = pz,$$

and let the plane to which the generatrix is parallel, be the plane of XZ. The equation of the generatrix, on the axis of Y, and parallel to XZ, is

(B) $$x = az, \qquad y = e.$$

Eliminating x,y,z between (A) and (B), we have,

(C) $$ae = me + pn.$$

Eliminate a and e between (B) and (C), and we have for the surface required,

$$yx = mzy + npz.$$

Ex. 2.—Find the twisted surface generated by a line moving parallel to ZX, one end of it on the axis of Y, and the other on the curve,

$$y^2 = 4mz, \qquad x = n.$$

By the procedure of Proposition LXXX., one or all of the functions of a may be eliminated from (497).

APPENDIX

TO THE

DIFFERENTIAL CALCULUS.

PROPOSITION A.

Determine the circle which touches three given curves.

Let the equation of the circle be

(1) $$(x - a)^2 + (y - \beta)^2 = R^2,$$

and let the three curves be represented by

(2) $$y' = \phi' x',$$

(3) $$y'' = \phi'' x'',$$

(4) $$y''' = \phi''' x'''.$$

Where (1) touches (2), (3), (4), respectively, it becomes

(5) $$(x' - a)^2 + (y' - \beta)^2 = R^2,$$

(6) $$(x'' - a)^2 + (y'' - \beta)^2 = R^2,$$

(7) $$(x''' - a)^2 + (y''' - \beta)^2 = R^2.$$

Let p be the differential coefficient of the circle (5), (6), or (7), and p', p'', p''', the differential coefficients of (2), (3), (4), respectively. Then, since the circle touches each of the curves (2), (3), (4), we have,

(8) $$p = p',$$

(9) $$p = p'',$$

(10) $$p = p'''.$$

Eliminate x', y', from (2), (5), and (8), and we may represent the result by

(11) $$F(a, \beta, R) = o.$$

Eliminate x'' and y'' from (3), (6), and (9), and we have,

(12) $$F'(a, \beta, R) = o.$$

Eliminate x''', y''' from (4), (7), and (10), and we have,

(13) $$F''(a, \beta, R) = o.$$

Solve the three equations (11), (12), (13), for a, β, and R, and these values determine the circle in position and magnitude.

This Proposition shows how the Calculus may be applied to Geometrical propositions in which the curve (in this case the circle), is limited to one curve of the species.

Cor. If the circle pass through one point $(m,n,)$ and touch two curves, viz. (2) and (3), equation (7) would become

$$(m - a)^2 + (n - \beta)^2 = R^2, \tag{14}$$

which takes the place of one of the relations, (11), (12), (13). In a similar manner, if the circle passes through two points, and touches one curve, the condition of passing through two points, furnishes two of the equations (11), (12), (13). The condition of touching a given curve (2) furnishes the other, and the circle is determined.

We will generalise this process in the next Proposition.

PROPOSITION B.

A curve containing a given number of parameters touches as many curves as it has parameters, determine the curve.

It has already been observed that a curve or surface may be subjected to as many conditions as it contains parameters. Hence, a curve containing n parameters, is fixed by subjecting it to n conditions—such as touching n curves—or passing through n points—or passing through m points, and touching $n - m$ lines or touching $n—m$ curves, and having m parameters of a given magnitude, &c. Hence, if the required curve has n parameters and touches n curves, each of the differential coefficients of the n curves is equal to the differential coefficient of the required curve. This furnishes n equations. The n curves furnish n more equations, and the required curve taken n times, i. e. once for each point of tangency furnishes n more equations. Hence, we have $3n$ equations, from which if we eliminate the $2n$ co-ordinates, we have n equations among the n parameters, from which they may be determined, and this fixes the proposed curve.

Ex.—Let the curve be the ellipse.

The equation of the ellipse in its general form is

(15) $$a^2 (y - n)^2 + b^2 (x - m)^2 = a^2 b^2.$$

This contains four parameters, a, b, m and n. Hence, it may be subjected to touch four given curves. Let

(16) $$y = fx, \quad y = f'x, \quad y = f''x, \quad y = f'''x,$$

be the four curves to which (15) is tangent.

Then, if p be the differential coefficient of (15), and p', p'', p''', p'''', be the differential coefficients of (16) respectively, we have,

(17) $$p = p', \quad p = p'', \quad p = p''', \quad p = p''''.$$

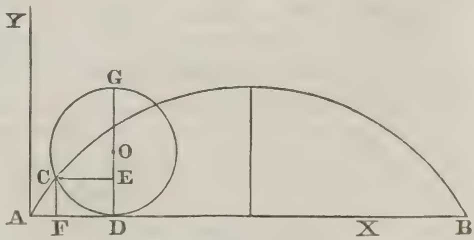
Since (15) touches each of the curves (16), we may take (15) as four different equations, according as it touches the first, second, third, or fourth of (16). Hence (15), (16), (17) are twelve equations, from which we may eliminate the eight co-ordinates of the points of tangency, and we will have as the result, four equations, among the parameters a,b,m,n. By solving these four equations, these parameters may be determined, and the ellipse (15) described, so as to touch the four curves (16).

PROPOSITION C.

Determine the equation of the cycloid.

Definition.—If a circle roll along a given straight line AB, the curve described by a point C on the circle, is called the cycloid.

FIG. 73.



Let the point C coincide with the given line AB at A.

Take A as the origin, and the given line AB as the axis of X. Let $r = \text{OD}$, the radius of the generating circle. Then it is obvious that when the circle rolls along AB, every point in the circumference passes in contact with AB. Consequently AD = CD.

Let (x,y) be the point C, then

(18) $$AF = AD - FD,$$

and we obviously have,

(19) $$AD = CD = \text{versin.}^{-1}y, \text{ and}$$

(20) $$FD = CE = (2ry - y^2)^{\frac{1}{2}}.$$

These values put into (18), we have,

(21) $$x = \text{versin.}^{-1}y - (2ry - y^2)^{\frac{1}{2}}.$$

This is the equation of the cycloid.

If we differentiate (21), we have, after reducing the second side to a common denominator,

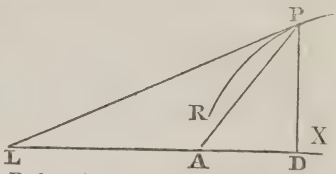
(22) $$dx = \frac{ydy}{(2ry - y^2)^{\frac{1}{2}}}.$$

By means of (21), or (22), the other properties of the curve can be determined.

PROPOSITION D.

Determine the angle which the radius vector makes with a given curve at a given point.

FIG. 74.



The angle which a line makes with a curve at any point is the same as the angle which the line makes with the tangent to the curve at that point.

Let PR be the given curve, and P the given point on it.

Let the origin A of rectangular axes be the pole, and the axis of X the angular axis. Let

$$AP = \rho, \quad PAX = \omega, \quad AD = x, \quad PD = y.$$

Draw PL tangent to the curve at P, then LPA is the angle required. By (57), Differential Calculus, we have,

(a) $$\frac{dy}{dx} = \tan.L = p,$$

and by the figure we have,

(b) $$\tan.\omega = \frac{y}{x}.$$

Also by the figure we have,

$$(c) \qquad \tan.\mathrm{LPA} = \tan.(\mathrm{PAX} - \mathrm{PLX}).$$

Expand (c), and substitute into the expansion the values of the angles in (a) and (b) and we have,

$$(d) \qquad \tan.\mathrm{LPA} = \frac{ydx - xdy}{xdx + ydy}.$$

But in the figure we have,

$$(e) \qquad x = \rho \cos.\omega, \qquad y = \rho \sin.\omega.$$

Differentiate (e), and eliminate x,y, dx, and dy, from (d), by means of (e) and the differentials of (e), and we have,

$$(23) \qquad \tan.\mathrm{LPA} = \frac{\rho d\omega}{d\rho}.$$

This is the value of the tangent of the angle required.

If the equation of PR be

$$(f) \qquad \rho = \phi\omega.$$

we put the value of $d\omega \div d\rho$, deduced from (f), into (23) and we have the tangent of the angle in terms of the polar co-ordinates.

By means of the trigonometrical relation,

$$(24) \qquad \tan. = \frac{\sin.}{\cos.},$$

we can, by using the values of the tangent of the angle in (23), determine the values of the sine, cosine, secant, &c., of the angle LPA, viz.

$$(25) \qquad \sin.\mathrm{LPA} = \frac{\rho d\omega}{(d\rho^2 + \rho^2 d\omega^2)^{\frac{1}{2}}},$$

$$(26) \qquad \cos.\mathrm{LPA} = \frac{d\rho}{(d\rho^2 + \rho^2 d\omega^2)^{\frac{1}{2}}}, \ \&.$$

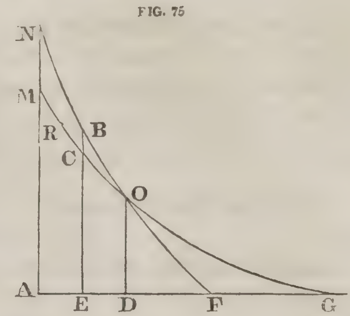
Ex.—Let the curve PR be $\rho = a\omega^2$, find the sine, cosine, and tangent of the angle LPA.

PROPOSITION E.

SUPPLEMENTARY TO PROPOSITION XXXVII.

Determine the locus of the intersection of two consecutive lines.

FIG. 75



Let the equation of MOG be

(27) $\varphi(x,y,\beta) = o = u.$

This line, [whether straight or curved], evidently depends for its position on the parameter β, which enters its equation (27). If we differentiate (27) for x and y variable, and β constant, we have,

(28) $$\frac{du}{dx}\frac{dx}{} + \frac{du}{dy}\frac{dy}{} = o.$$

If we differentiate it for y and β variable, and x constant, we have,

(29) $$\frac{du\,dy}{dy} + \frac{du\,d\beta}{d\beta} = o.$$

The differential (28) may be regarded as the passing from a point C to the consecutive point R on the same curve, and (29) as the passing on the ordinate EB from a point C to a point B on the consecutive curve. It is obvious at the point O, where the consecutive curves intersect that both x and y remain constant in (27), while β varies. Hence, if y be constant in (29), dy is zero and that equation becomes for the point O,

(30) $$\frac{du\,d\beta}{d\beta} = o,$$

and we have (27) and (30), to eliminate β, the result will be of the form,

(31) $$\psi(x, y) = o,$$

and denote the locus of the point of intersection.

Cor. The curve (27) is tangent to (31).

For the x and y of (27) are the x and y of (31) at the point of intersection O, and if (27) be differentiated for x, y, and β variable, we have,

(32) $$\frac{du\,dx}{dx} + \frac{du\,dy}{dy} + \frac{du\,d\beta}{d\beta} = o,$$

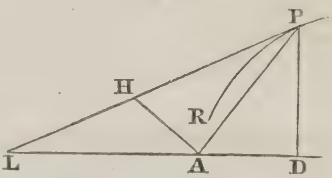
which, when β is determined by (30), is the differential equation of

(31). But the condition (30) reduces (32) to (28). Hence, the differential coefficients of (27) and (31) being the same, the curves are tangent to each other.

PROPOSITION F.

Determine the length of the polar subtangent of a given curve.

FIG. 76.



Draw PH tangent to the curve at P. Let A be the pole.

Draw AH perpendicular to the radius vector AP.

In the right angled triangle PAH, we have, by Trigonometry.

(33) $$AH = AP. \tan. HPA.$$

Put ρ for the radius vector AP, and putting into (33) for tan. HPA, its value at (23), Proposition D we have,

(34) $$AH = \rho^2 \frac{d\omega}{d\rho},$$

which agrees with the result in Proposition XXXIII.

By means of Propositions D and F, we deduce the values of the polar subtangent, subnormal, &c., without having recourse to the consideration of indefinitely small quantities.

THE

INTEGRAL CALCULUS.

CHAPTER I.

PRINCIPLES OF INTEGRATION.

THE Integral Calculus is the inverse of the Differential, and teaches the method of returning from a given differential to the primitive function from which it is derived. This process is termed Integration. The Rules of procedure in this Calculus are therefore to be obtained by inverting those of the Differential Calculus.

The first rule of the Differential Calculus teaches the method of differentiating a monomial raised to a power. The inverse of this is the first rule of the Integral Calculus.

RULE I.

To integrate a monomial, increase the exponent by unity, divide by this increased exponent, and by the differential of the variable.

The integral of $x^m dx$ is, by this rule, $\dfrac{x^{m+1}}{m+1}$, which is made by increasing the exponent by one, and dividing by $m + 1$, the increased exponent, and by dx. The differentiation of this last expression produces the first. Hence we may say, generally, that the object of integration is to obtain an expression whose differential is the differential proposed.

The symbol $\int$ is employed as the reverse of the character d. It is called the sign of integration, and denotes that the quantity before which it is placed is to be integrated. Thus $\int x^n dx$

denotes that $x^n dx$ is to be integrated. This symbol is the first letter of the word *sum*, and was used by the first writers on the Calculus, to express that integration was the summing, or adding together of the indefinitely small quantities of which the integrated expression is composed. Thus if y represent a straight line, dy is an indefinitely small portion of that line, and to integrate dy is to sum or add together these small portions. Hence the integral of dy is found by simply removing the characteristic d from dy.

According to Rule II. of the Differential Calculus, a constant quantity added to a given function disappears in differentiation.

Inversely we have,

RULE II.

In integrating, a constant quantity must be added to the integral.

Thus the complete integral of $x^n dx$ is,

$$\text{(1)} \qquad \int x^n dx = \frac{x^{n+1}}{n+1} + \text{C}.$$

where C designates the constant that may have disappeared in differentiation. The method of determining this constant will be given hereafter.

By Rule III. of the Differential Calculus, in differentiating the product of a constant factor and a variable, the constant factor remains. Inversely we have,

RULE III.

In integrating, if there be a constant factor to the differential, it remains.

Consequently, in such an expression as $ax^n dx$, we may put the factor a before the sign of integration. Thus,

$$\text{(2)} \qquad \int ax^n dx = a \int x^n dx = \frac{ax^{n+1}}{n+1} + \text{C}.$$

By Rule IV. of the Differential Calculus, the differential of the sum of any number of variables is the sum of their differentials. Inversely we have,

RULE IV.

The integral of the sum of any number of differentials is found

by integrating each term separately, and adding the terms together.

Thus if we have,

(3) $$dy = ax^2dx + bx\,dx - x^4dx,$$

the integral found by integrating each term is,

(4) $$\therefore\ y = \frac{ax^3}{3} + \frac{bx^2}{2} - \frac{x^5}{5} + \text{C}.$$

These Rules apply to the cases where each term to be integrated contains one variable, and no more. For sake of convenience, we will sometimes put our examples in the form of equations, taking only that form of equation in which the first side is a simple differential.

As examples under the foregoing Rules, take the following.

$$dy = adx + x^{\frac{1}{2}}dx \quad \therefore \quad y = ax + \frac{2}{3}x^{\frac{3}{2}} + \text{C}.$$

$$dy = \frac{adx}{x^4} = ax^{-4}dx \quad \therefore \quad y = -\frac{a}{3x^3} + \text{C}.$$

$$dy = \frac{adx}{x^n} + bx^{\frac{1}{n}}dx \quad \therefore \quad y = \frac{ax^{-n+1}}{-n+1} + \frac{nbx^{\frac{1+n}{n}}}{1+n} + \text{C}.$$

$$dy = ax^{\frac{n}{m}}dx + x^{\frac{3}{4}}dx + dx.$$

In these examples, we observe that if the variable in the monomial be in the denominator, it must, (with an exception to be presently noticed), be brought to the numerator before integrating.

In the Differential Calculus, (19), we observed, that a polynomial, as

(5) $$y = (a + bx^4)^n,$$

could be differentiated as a monomial by regarding the part within the vinculum as a single quantity raised to the power n. The differential of (5) is

(6) $$dy = n\,(a + bx^4)^{n-1}\,4\,bx^3\,dx.$$

To produce (5) from (6), we must obviously increase the exponent of the binomial by unity, divide by this increased exponent n, and by $4bx^3\,dx$, which is the differential of the part within the vinculum. Hence,

RULE V.

A polynomial may be integrated as a monomial if the part without the vinculum be the differential of the part within.

In the two following examples the part without the vinculum is the complete differential of the part within.

Ex. 1. $dy = (a + bx)^n \, bdx, \quad \therefore \; y = \dfrac{(a + bx)^{n+1}}{n + 1} + C.$

Ex. 2. $dy = (a + bx^2)^{-\frac{1}{2}} \, 2\,bx\,dx \quad \therefore \; y = 2\,(a + bx^2)^{\frac{1}{2}} + C.$

If we had such an expression to integrate as,

(7) $$dy = (a + bx)^n \, dx,$$

where the part without the vinculum is the differential of the part within, except by a constant factor b, we may multiply by b the factor required, then integrate, and divide the integral by the factor. The integral of (7) is, by this process,

(8) $$y = \frac{(a + bx)^{n+1}}{b\,(n + 1)} + C.$$

It is usual to denote the integral of (7), thus

(9) $$y = \frac{1}{b} \int (a + bx)^n \, bdx,$$

where, by multiplying the part after the sign of integration by b, and putting the reciprocal of b before the sign of integration, the expression remains unchanged in value, and the part outside the vinculum is the complete differential of the part within. If the part outside the vinculum requires a variable factor to make it the differential of the part within, the binomial cannot be integrated by the foregoing process. Thus, if we have,

(10) $$dy = (a + bx^3)^n \, xdx,$$

a variable factor is required to make the part without the vinculum, the differential of the part within, and (10) is not immediately integrable by Rule V.

If we have such an expression as,

(11) $$dy = \frac{dx}{x} \quad \therefore \; dy = x^{-1} dx.$$

If this be integrated by Rule I, we have, $\therefore y =$ infinity, and the integration fails.

But the numerator of (11) is the differential of the denominator, and by Rule V of the Differential Calculus, we know that this came from differentiating the logarithm of the denominator. The integral of (11) is, therefore,

(12) $$y = \log.x + C.$$

Hence, we have,

RULE VI.

If the numerator be the differential of the denominator, the integral is the logarithm of the denominator.

Ex. $$dy = \frac{dx}{a + x} \quad \therefore y = \log.(a + x) + C.$$

Ex. 2. $$dy = \frac{2bx\,dx}{a + bx^2} \quad \therefore y = \log.(a + bx^2) + C.$$

If the numerator fail by a constant factor of being the differential of the denominator, multiply by this factor, then integrate, and divide the integral by the factor. If, for example, we have,

(13) $$dy = \frac{dx}{a + bx},$$

here the numerator, is the differential of the denominator, except by the constant factor b. Then, multiplying by this factor b, integrating and dividing by b, we have,

(14) $$y = \frac{1}{b}\log.(a + bx) + C.$$

As another example, take

$$dy = \frac{xdx}{a - x^2},$$

here the numerator wants the factor—2 to be the differential of the denominator. Hence, the integral is

$$\therefore y = -\frac{1}{2}\log.(a - x^2) + C.$$

If we put $C = \log. c$, (12) becomes

(15) $$y = \log.x + \log.c = \log.cx.$$

If e be the number whose log. is unity, (15) may be written

(16) $$e^y = cx,$$

for the log. of (16) is (15).

These essential principles will be employed hereafter.

As an illustration, take the following examples.

$$dy = \frac{x^2dx}{a-bx^3} \quad \therefore \quad y = -\frac{1}{3b}\log.(a-bx^3) + \text{C}.$$

$$dy = \frac{xdx}{a+bx^2} \quad \therefore \quad y = \frac{1}{2b}\log.(a+bx^2) + \text{C, or}$$

$$y = \frac{1}{2b}\log.c\,(a+bx^2), \qquad \text{or,}$$

$$e^y = \left(ca + cbx^2\right)^{\frac{1}{2b}}.$$

$$dy = \frac{x^{n-1}dx}{a-bx^{n-1}}, \qquad dy = \frac{(a+2x)\,dx}{ax+x^2}.$$

INTEGRATION BY CIRCULAR ARCS.

We have seen in Differential Calculus, Proposition XXXII., that if we have,

(17) $$y = \sin.^{-1}x,$$

then by differentiating we have,

(18) $$dy = \frac{dx}{(1-x^2)^{\frac{1}{2}}}.$$

Consequently (17) is the integral of (18), i. e.,

(19) $$\int\frac{dx}{(1-x^2)^{\frac{1}{2}}} = \sin.^{-1}x.$$

If we had the form,

$$\frac{adx}{(b^2-c^2x^2)^{\frac{1}{2}}},$$

then putting $c = bm$, it becomes,

$$\frac{a}{bm}\cdot\frac{mdx}{(1-m^2x^2)^{\frac{1}{2}}},$$

and the integral is,

$$(20) \qquad \frac{a}{bm}\int \frac{mdx}{(1 - m^2x^2)^{\frac{1}{2}}} = \frac{a}{bm} \sin.^{-1} mx + C,$$

or restoring the value of m,

$$(21) \qquad a\int \frac{dx}{(b^2 - c^2x^2)^{\frac{1}{2}}} = \frac{a}{c} \sin.^{-1} \frac{cx}{b} + C.$$

This will serve as a form for integrating all similar expressions. We observe in (20) that the sine line is mx, and the radius unity.

Again, by Differential Calculus, Proposition XXXII.,

$$(22) \qquad d.\cos.^{-1} x = \frac{-dx}{(1 - x^2)^{\frac{1}{2}}}.$$

Consequently,

$$(23) \qquad \int \frac{-dx}{(1 - x^2)^{\frac{1}{2}}} = \cos.^{-1} x.$$

If we had the form,

$$\frac{-adx}{(b^2 - c^2x^2)^{\frac{1}{2}}},$$

putting $c = bm$, then integrating and restoring the value of m, we have,

$$(24) \qquad a\int \frac{-dx}{(b^2 - c^2x^2)^{\frac{1}{2}}} = \frac{a}{c} \cos.^{-1} \frac{cx}{b} + C.$$

By the same Proposition of the Differential Calculus, we have,

$$(25) \qquad d.\tan.^{-1} u = \frac{du}{1 + u^2}$$

Integrating, we have,

$$(26) \qquad \therefore \int \frac{du}{1 + u^2} = \tan.^{-1} u.$$

If we had the form,

$$\frac{a\,du}{b^2 + c^2u^2}$$

then putting $c = mb$, this reduces to

$$\frac{a}{b^2m} \cdot \frac{m\,du}{1 + m^2u^2}$$

Since mu may be regarded as the tangent line, this may be integrated by comparison to (26), and we have,

$$(27)\qquad \frac{a}{b^2m}\int\frac{m\,du}{1+m^2u^2}=\frac{a}{b^2m}\cdot\tan.^{-1}mu.$$

Restoring the value of m, (27) becomes,

$$(28)\qquad a\int\frac{du}{b^2+c^2u^2}=\frac{a}{bc}\tan.^{-1}\frac{cu}{b}+\text{C}.$$

which will serve as a form for integrating all similar expressions.

As an example, take the equation.

$$(d)\qquad dy=\frac{dx}{4+x^2}.$$

Comparing this with (28), we have $b=2$, $c=1$, and (d) becomes,

$$(e)\qquad \therefore\ y=\frac{1}{2}\tan.^{-1}\frac{x}{2}+\text{C}.$$

Again, by Proposition XXXII., Differential Calculus, we have

$$(29)\qquad d.\text{versin.}^{-1}v=\frac{dv}{(2v-v^2)^{\frac{1}{2}}}.$$

Consequently,

$$(30)\qquad \int\frac{dv}{(2v-v^2)^{\frac{1}{2}}}=\text{versin.}^{-1}v.$$

If we had the form,

$$(31)\qquad dy=\frac{dx}{(b^2x-c^2x^2)^{\frac{1}{2}}},$$

assume $\dfrac{2c^2x}{b^2}=v$, or, $x=\dfrac{b^2v}{2c^2}$.

Put this value of x into (31), and it becomes,

$$(32)\qquad dy=\frac{1}{c}\cdot\frac{dv}{(2v-v^2)^{\frac{1}{2}}},$$

and integrating by (30), we have,

$$(33)\qquad y=\frac{1}{c}\,\text{versin.}^{-1}v+\text{C}.$$

Restoring in this the value of v, we have,

$$(34)\qquad \int\frac{dx}{(b^2x-c^2x^2)^{\frac{1}{2}}}=\frac{1}{c}\,\text{versin.}^{-1}\frac{2c^2x}{b^2}+\text{C}.$$

We can integrate many expressions by the form (34).

As an example, take the following:

$$(f) \qquad dy = \frac{adx}{(3x - mx^2)^{\frac{1}{2}}}.$$

Comparing this with (34), we have, $b^2 = 3$, $c^2 = m$, and (f) becomes,

$$\therefore \quad y = \frac{a}{\sqrt{m}} \text{ versin.}^{-1} \frac{2mx}{3} + \text{C}.$$

Integrate the expressions

$$dy = \frac{dx}{(3 - x^2)^{\frac{1}{2}}}, \quad dy = \frac{dx}{2 + 5x^2}, \quad dy = \frac{dx}{(7x - 5x^2)^{\frac{1}{2}}}.$$

CHAPTER II.

INTEGRATION OF RATIONAL FRACTIONS.

A rational fraction is one that contains no surd quantity.

Lemma.

Divide a rational fraction into several partial fractions.

This is a procedure of Algebra; but as it is not much treated of in books on that subject, it may be well to explain it here. Let the fraction be

$$(35) \qquad \frac{ax}{x^2 - 2x - 3}.$$

Any such fraction can be divided into as many partial fractions as there are factors in its denominator. To find these factors, put the denominator equal to zero, that is, assume the equation

$$(36) \qquad x^2 - 2x - 3 = o.$$

Find the roots of this equation, which are $x = 3$, and $x = -1$; hence, the factors of (36) are $x - 3$, and $x + 1$.

Now assume the equation

$$(37) \qquad \frac{ax}{x^2 - 2x - 3} = \frac{\text{A}}{x - 3} + \frac{\text{B}}{x + 1},$$

where A and B are unknown quantities not containing x, and the

denominators of the assumed fractions are the factors of the denominator of the given fraction. Determine A and B by the method of indeterminate coefficients, which is done by reducing (37) to monomial terms, by clearing of fractions. This gives

(38) $$ax = Ax + A + Bx - 3B.$$

According to the method of indeterminate coefficients, the coefficients of the like powers of x in (38) are equal. Hence, we have,

(39) $$a = A + B, \quad \text{and} \quad o = A - 3B.$$

Solve these two equations for A and B, and we have,

$$A = \frac{3a}{4}, \quad \text{and} \quad B = \frac{a}{4},$$

these values put into (37), we have,

(40) $$\frac{ax}{x^2 - 2x - 3} = \frac{3a}{4(x-3)} + \frac{a}{4(x+1)},$$

and the fraction is divided into two partial fractions.

This procedure supposes the highest power of x in the denominator to exceed at least by unity, the highest power of x in the numerator. If we had such a fraction as $\frac{x^3}{x^2 - b^2}$ by dividing the numerator by the denominator this may be put in the form,

$$x + \frac{b^2x}{x^2 - b^2},$$

where the last term only is a rational fraction.

If some, or all of the factors of the denominator are equal, the process must be somewhat modified. Let the fraction be

$$\frac{bx + x^2}{(x+1)(x-1)^3},$$

where three of the factors of the denominator are equal. If we were to divide this as was done in (37), the division would fail, as will be found upon trial. But we may assume,

(41)
$$\frac{bx + x^2}{(x+1)(x-1)^3} = \frac{A}{x+1} + \frac{B}{(x-1)^3} + \frac{C}{(x-1)^2} + \frac{D}{x-1},$$

where the second side shows the form of the partial fractions into

which the given fraction can be resolved. The values of the constants A, B, C, D, in (41) are found as in (37).

If there be equal sets of roots in the denominator of the proposed fraction, as in the form,

$$\frac{ax}{(x-a)^2(x+b)^2},$$

we may divide it as follows. Assume

(42)
$$\frac{ax}{(x-a)^2(x+b)^2} = \frac{A}{(x-a)^2} + \frac{B}{x-a} + \frac{C}{(x+b)^2} + \frac{D}{x+b},$$

where the constants A, B, C, D, may be be found, as in (37).

If some or all of the factors of the denominator be imaginary, the process requires another modification. If we have,

(43) $$\frac{ax}{(b+x)(x^2+2mx+m^2+n^2)},$$

the denominator being equated to zero, shows the factors

$$b+x, \qquad x+m-n\sqrt{-1}, \quad \text{and} \quad x+m+n\sqrt{-1},$$

where two of the factors are imaginary. And we may observe, in general, that the Theory of Equations shows that imaginary factors always occur in pairs. Before dividing the form (43) into partial fractions, we may observe that the second factor of the denominator may be put into the form,

$$(x+m)^2+n^2,$$

and if $x+m$ be put $=z$, this is simply z^2+n^2, so that any form (43) may always be reduced to the simpler form,

(44) $$\frac{ax}{(b+x)(x^2+n^2)},$$

If the second factor in the denominator of (43) were, for example, x^2+x+1, this is the same as $(x+\frac{1}{2})^2+\frac{3}{4}$, and if we assume $x+\frac{1}{2}=z$, it becomes $z^2+\frac{3}{4}$, which agrees in form with the second factor of the denominator of (44). To divide (44) into partial fractions is therefore the same in effect as to divide (43). To divide (44), put

(45) $$\frac{ax}{(b+x)(x^2+n^2)} = \frac{A}{b+x} + \frac{Cx}{x^2+n^2} + \frac{D}{x^2+n^2},$$

where A, C, D may be found, as in (37), by the method of indeterminate coefficients.

If we had several sets of imaginary factors in the denominator, the process would be as follows. Let

$$\frac{ax^2}{(x^2+n^2)^2},$$

be the fraction. Assume the equation

$$(46) \qquad \frac{ax^2}{(x^2+n^2)^2} = \frac{Ax}{(x^2+n^2)^2} + \frac{Bx}{x^2+n^2} + \frac{C}{(x^2+n^2)^2} + \frac{D}{x^2+n^2}.$$

Here A, B, C, D may be determined by the method of indeterminate coefficients, as was done at (37).

From the foregoing, we observe, that there are four classes of rational fractions, which may be divided into partial fractions.

1st.—*When the factors of the denominator are real and unequal, as in* (37).

2d.—*When the factors of the denominator are real, and some or all of them equal, as in* (41), (42).

3d.—*When the denominator contains one set of imaginary factors, as in* (45).

4th.—*When the denominator contains more than one set of imaginary factors, as in* (46).

As a generalisation of the first of these classes, suppose we had the fraction $\dfrac{ax}{pqrs\ldots}$, where $p,q,r,s,$ &c., are the factors of the denominator, and functions of x. Then we assume

$$(46a) \qquad \frac{ax}{pqrs\ldots} = \frac{A}{p} + \frac{B}{q} + \frac{C}{r} + \frac{D}{s}.$$

Here A, B, C, D are to be determined by the method of indeterminate coefficients.

As a generalisation of the second class, suppose we had the fraction $\dfrac{ax}{P^n}$, where P is a function of x. Assume this fraction equal to the several partial fractions, as follows.

$$(46b) \qquad \frac{ax}{P^n} = \frac{A}{P^n} + \frac{B}{P^{n-1}} + \frac{C}{P^{n-2}} + \frac{D}{P^{n-3}} + \ldots \frac{M}{P},$$

As a generalisation of the fourth class, put $D = x^2 + n^2$, and suppose we have the fraction $\frac{ax^2}{D^c}$. Assume this fraction equal to several partial fractions, as follows.

$$(46c) \quad \left\{ \begin{aligned} \frac{ax^2}{D^c} &= \frac{Ax}{D^c} + \frac{Bx}{D^{c-1}} + \dots \frac{Mx}{D} \\ &+ \frac{A'}{D^c} + \frac{B'}{D^{c-1}} + \dots \frac{M'}{D}. \end{aligned} \right.$$

Determine A, B, . . . M, A′, B′, ...M′, by the method of indeterminate coefficients, and the second side of (46c) exhibits the partial fractions into which the first side may be divided.

We will treat of the integration of each of the four classes of fractions now discussed.

Case 1.

Integrate a rational fraction when the factors of the denominator are real and unequal.

Divide the rational fraction into its partial fractions, and integrate each partial fraction separately.

Take the fraction

$$\frac{ax\,dx}{x^2 - 2x - 3}.$$

Omitting the dx, this is divided into its partial fractions at (40). Hence multiply (40) by dx, and integrating each term of the second side, we have for the integral required,

$$(47) \quad a\int \frac{dx}{x^2-2x-3} = \frac{3a}{4}\log.(x-3) + \frac{a}{4}\log.(x+1) + C.$$

Ex. 2. Take the fraction

$$(48) \quad dy = \frac{ax\,dx.}{x^3 - 2x^2 - 3x}.$$

Here the factors of the denominator are x, $x - 3$, $x + 1$. Then we may assume

$$(49) \quad \frac{ax}{x^3 - 2x^2 - 3x} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+1}.$$

Multiply (49) by dx, and integrating each term of the second side, we have the integral of (48), viz.

(50) $y = \text{A} \log.x + \text{B} \log.(x - 3) + \text{C} \log.(x + 1) + \text{C}'$,

where A, B, C may be determined from (49), as in (37).

Ex. 3. Integrate the equations,

$$dy = \frac{adx}{x^2 - a^2}, \qquad dy = \frac{(2x + 3)\,dx}{x^3 + x^2 - 2x}.$$

Case 2.

Integrate a rational fraction when the factors of the denominator are real, and some or all of them equal.

Divide the rational fraction into its partial fractions, and integrate each fraction separately.

Thus if we multiply (41) by dx, the first side is a rational fraction, whose integral is the sum of the integrals of the fractions on the second side, each of which is obviously integrable, the first and last term being logarithms, and the second and third coming under Rule V. In like manner, if we multiply both sides of (42) by dx, each fraction on the second side is integrable by Rule V. or VI. Their sum is the integral of the first side.

Case 3.

Integrate a rational fraction when one set of imaginary factors enters the denominator.

Divide the fraction into its partial fractions, and integrate each partial fraction separately.

Thus if we multiply (45) by dx, the first side is a rational fraction, whose integral is the sum of the integrals on the second side. The two first terms on the second side are integrated by Rule VI., and the third term by (28). Hence,

$$(51) \qquad \left\{ \begin{aligned} a\int \frac{xdx}{(b + x)\,(x^2 + n^2)} &= \text{A} \log.(b + x) \\ &+ \frac{\text{C}}{2} \log.(x^2 + n^2) + \frac{\text{D}}{n} \tan.^{-1} \frac{x}{n} + \text{C}. \end{aligned} \right.$$

Ex.—Integrate the equations,

$$dy = \frac{(a + bx)\,dx,}{x^3 - 1}, \qquad dy = \frac{dx}{x^3 + x}.$$

Case 4.

Integrate a rational fraction when several sets of imaginary factors enter the denominator.

Divide the fraction into its partial fractions, and integrate each partial fraction separately.

Thus if we multiply (46) by dx, the integral of each term on the second side is readily effected, except the third term. The first belongs to Rule V., the second to Rule VI., and the fourth to (28). The third term, viz :

$$\frac{C\,dx}{(x^2+n^2)^2}, \tag{52}$$

may be reduced to depend for its integral upon the integral of the form $\dfrac{D\,dx}{x^2+n^2}$, by diminishing the exponent of the denominator by unity. But as a simple formula for reducing (52) will be given in a more general process hereafter, we will not here give the integral of the form (52).

Thus by dividing a rational fraction into its partial fractions, we may integrate an extensive class of differentials by the simple rules given in the first Chapter.

CHAPTER III.

INTEGRATION OF IRRATIONAL FRACTIONS.

An irrational fraction is one that contains the variable under a fractional exponent.

If the fraction consists of monomials only, it may be readily rationalised, and will then be brought under some of the processes of rational fractions. Take as an example,

$$dy = \frac{(x^{\frac{1}{4}} - m)\,dx}{x^{\frac{1}{2}} - x^{\frac{1}{3}}}. \tag{53}$$

Assume x equal to z raised to such a power that the several fractional powers of x in (53) will be expressed in z with integer exponents. This may always be done by putting x equal to z raised to a power given by the common denominator of the fractional exponents, in the proposed term to be integrated. In (53) put

$$(54)\qquad \begin{cases} x = z^{12} \quad \therefore \quad x^{\frac{1}{2}} = z^6, \quad x^{\frac{1}{3}} = z^4, \quad \text{and} \\ x^{\frac{1}{4}} = z^3, \quad \text{also} \quad dx = 12z^{11}\,dz. \end{cases}$$

Put the values (54) of x in terms of z, into (53), and it becomes,

$$(55)\qquad dy = \frac{(z^{10} - mz^7)\,12dz}{z^2 - 1}.$$

In which, after dividing the numerator by the denominator, the integral will finally depend upon a rational fraction whose denominator is $z^2 - 1$.

If we have to integrate an irrational fraction of the form,

$$(56)\qquad \frac{dx}{(a + bx + cx^2)^{\frac{1}{2}}},$$

we must first rationalise the denominator, which may be done by some of the processes pointed out in the Diophantine Analysis. The procedure to rationalise such an example as (56), may be represented generally as follows.

Let us designate by $\varphi(\sqrt{x})$, any combination of x under the radical. If then we have to integrate the fraction

$$(57)\qquad \frac{dx}{\varphi(\sqrt{x})},$$

we may assume

$$(58)\qquad \varphi(\sqrt{x}) = \mathrm{F}\,(x,z),$$

where $\mathrm{F}\,(x,z)$ denotes such a combination of x,z and constant quantities as will, by the Diophantine Analysis, render the first side of (58) rational. Solve (58) for x, and we may represent the result by

$$(59)\qquad x = \pi z.$$

Differentiate this, and we have,

$$(60)\qquad dx = \pi' z\,dz.$$

Substitute (59) into (58) second side, and we get

$$(61)\qquad \varphi(\sqrt{x}) = \mathrm{F}\,(\pi z, z).$$

Put (61) and (60) into (57), and it becomes,

$$(62)\qquad \frac{\pi' z\,dz}{\mathrm{F}\,(\pi z, z)},$$

which is rational, and may be integrated by some of the preceding methods.

For the various modes of rationalising any given quantity, the student is referred to the Diophantine Analysis, whose principles and processes are explained in most books on Algebra.

As a particular example of (57) take

$$(a) \qquad \frac{dx}{(a + b^2x^2)^{\frac{1}{2}}}.$$

In this case (58) becomes,

$$(b) \qquad (a + b^2x^2)^{\frac{1}{2}} = bx + z.$$

Solve this for x, and we get

$$(c) \qquad x = \frac{a - z^2}{2bz}.$$

The differential of this is

$$(d) \qquad dx = -\frac{a + z^2}{2bz^2}.\ dz.$$

Put (c) into second side of (b), and we have,

$$(e) \qquad (a + b^2x^2)^{\frac{1}{2}} = \frac{a + z^2}{2z}.$$

Put (d) and (e) into (a), and it becomes,

$$(f) \qquad -\frac{1}{b}.\ \frac{dz}{z},$$

whose integral is

$$(g) \qquad -\frac{1}{b}\log.z + \mathrm{C}.$$

Restoring in (g) the value of z from (b) we have, for the integral of (a),

$$(63) \qquad \int \frac{dx}{(a + b^2x^2)^{\frac{1}{2}}} = -\frac{1}{b}\log.(\sqrt{a + b^2x^2} - bx) + \mathrm{C},$$

which might be put into a more simple form.

Before proceeding to rationalise (57), we may frequently put it into a more convenient form, by dividing the numerator and denominator by some of the constants that enter into the radical. Thus to rationalise (56), we might first divide the numerator and denominator by $\sqrt{c}$, or by $\sqrt{a}$, which would render the denominator readily rational. For, dividing by $\sqrt{c}$, and putting

$$\frac{a}{c} = m, \qquad \text{and} \qquad \frac{b}{c} = n,$$

the denominator of (56) is $(m + nx + x^2)^{\frac{1}{2}}$, which is rationalised by putting it $= x + z$.

If the denominator of (56) be divided by $\sqrt{a}$, then by putting

$$\frac{b}{a} = m, \qquad \text{and} \qquad \frac{c}{a} = n,$$

the denominator of (56) becomes $(1 + mx + nx^2)^{\frac{1}{2}}$, which is rationalised by putting it $= 1 + xz$.

This last is the most convenient form, if c in (56) be negative.

The student may integrate (56) for c positive and for c negative.

The same process would obviously apply if the denominator were of the form $\mathrm{F}x.\varphi\sqrt{x}$, for then the irrational part might be rationalised, and the rational part connected with it in terms of the new variable. Take for example, the equation,

$$(h) \qquad dy = \frac{dx}{(1 + x^2)(1 - x^2)^{\frac{1}{2}}}.$$

The irrational part of the denominator is rationalised by putting $(1 - x^2)^{\frac{1}{2}} = 1 - zx$, and equation (h) becomes,

$$(k) \qquad dy = \frac{2(z^2 + 1)\,dz}{z^4 + 6z^2 + 1},$$

which is a rational fraction in Case 4.

The denominator of (h) may also be rationalised by assuming

$$(l) \qquad (1 - x^2)^{\frac{1}{2}} = (1 - x)\,z,$$

which leads to a more convenient form than (k).

CHAPTER IV.

The object of this chapter is to integrate the Binomial Form.

$$(64) \qquad x^{m-1}dx\,(a + bx^n)^{p}.$$

Here m and n may be taken as whole numbers. For if they were fractional, we could, as in (53), by putting x equal to z, raised

to a power denoted by the common denominator of m and n, change (64) into a similar expression where the exponents corresponding to m and n would be integers.

We may also regard n in (64) as positive; for if it were negative, we could, by putting $x^{-n} = z^n$, change (64) to a similar expression, in which n would be positive. Nor need we have x in more than one term of the binomial (64). For if we had such a form as $x^{m-1}dx\,(ax^s + bx^n)^p$, this may be written $x^{ps+m-1}dx\,(a + bx^{n-s})^p$, which is the same in form as (64). If then we can integrate (64) when p is either whole or fractional, positive or negative, any of the other forms reducible to the form (64) may be integrated.

To integrate (64), put

$$(a) \qquad a + bx^n = z, \quad \therefore \quad x = \left(\frac{z-a}{b}\right)^{\frac{1}{n}} \quad \therefore\ x^m = \left(\frac{z-a}{b}\right)^{\frac{m}{n}},$$

and

$$(b) \qquad x^{m-1}dx = \frac{1}{n}\left(\frac{z-a}{b}\right)^{\frac{m}{n}-1} dz.$$

The values from (a) and (b) put into (64), it becomes,

$$(65) \qquad \frac{1}{n}\left(\frac{z-a}{b}\right)^{\frac{m}{n}-1} z^p dz.$$

If now $\dfrac{m}{n}$ be a whole number, or zero, the binomial of (65) is rational, and (65) may be integrated by some of the preceding methods.

Hence the binomial (64) *may be integrated if the exponent of the variable without the vinculum increased by unity is divisible by the exponent within.*

This is the first condition of integrability.

Take as an example,

$$(c) \qquad dy = x^5dx\,(a + bx^2)^{\frac{1}{4}}.$$

Here $m = b$, $n = 2$, $p = \frac{1}{4}$. The first condition of integrability is fulfilled, and (65) becomes,

$$(d) \qquad dy = \frac{1}{3}\, b^{2}\,(z - a)^2\, z^{\frac{1}{4}} dz,$$

where the binomial may be squared, and $z^{\frac{1}{2}}dz$ multiplied into it, and each term integrated separately.

If the first condition of integrability be not fulfilled, we can get another condition, as follows.

Divide (64) by x^{pn}, and then multiplying by x^{pn}, it becomes,

(66) $$x^{m+pn-1}dx\ (ax^{-n} + b)^p.$$

Put $z = ax^{-n} + b$, and by means of this, eliminate x and dx from (66), and it becomes,

(66*a*) $$-\frac{z^p}{na}\ dz \left(\frac{z-b}{a}\right)^{-\frac{m}{n}-p-1}.$$

This is a rational binomial if $\dfrac{m}{n} + p$ be a whole number.

Hence the binomial may be integrated if the exponent of the variable, without the parenthesis increased by unity and divided by the exponent within, and this quotient increased by the exponent of the parenthesis is a whole number. This is the second condition of integrability.

Ex.—Integrate the expression,

(*e*) $$x^{-2}\ (a + x^3)^{-\frac{5}{3}}\ dx.$$

Here $m - 1 = -2$, $n = 3$, $p = -\frac{5}{3}$, and the second condition of integrability is fulfilled, and putting

(*f*) $$z = ax^{-3} + 1,$$ (*e*) becomes

(*g*) $$-\frac{dz}{3a^2}\ \frac{(z-1)}{z^{\frac{5}{3}}} = -\frac{dz}{3a^2z^{\frac{2}{3}}} + \frac{dz}{3a^2z^{\frac{5}{3}}}.$$

These fractions are immediately integrable by Rule I.

CHAPTER V.

INTEGRATION BY PARTS.

By the Differential Calculus, we have,

(67) $$d.uv = u\,dv + v\,du.$$

From which, by integrating, we get

(68) $$\int u\,dv = vu - \int v\,du.$$

This is called the formula for integration by parts, and is of extensive application. By it any integral of the form $\int u\, dv$ is reduced to depend upon another of the form $\int v\, du$. The use of the formula will be best shown by a few examples.

Ex. 1. Integrate the expression,

(a) $$xdx.\log.x.$$

Divide this into the two parts, xdx and $\log.x$. Put

(b) $$u = \log.x, \quad \text{and} \quad dv = xdx.$$

Then differentiating the first, and integrating the second of (b), we have,

(c) $$du = \frac{dx}{x}, \quad \text{and} \quad v = \frac{x^2}{2}.$$

Put the values (b) and (c) into (68), and we get

(d) $$\int xdx.\log.x = \frac{x^2}{2}\log.x - \int\frac{xdx}{2}.$$

Here the last term is immediately integrable. Integrating i., we have, for the integral of (a),

(e) $$\int xdx.\log.x = \frac{x^2}{2}.\log.x - \frac{x^2}{4} + C.$$

Ex. 2. Integrate the expression,

(f) $$\sin.^{-1}x.\, dx. \qquad \text{Put}$$

(g) $$u = \sin.^{-1}x, \quad \text{and} \quad dv = dx.$$

Differentiate the first, integrate the second of (g), and substituting into (68), we have,

(h) $$\therefore \int \sin.^{-1}x\, dx = \sin.^{-1}x - \int (1 - x^2)^{-\frac{1}{2}}\, x\, dx,$$

where the last term is immediately integrable by Rule V. Integrating it, we have the integral of (h).

Ex. 3. Integrate the expression,

(k) $$\tan.^{-1}x.dx.$$

Put $\tan.^{-1}x = u$, and $dx = dv$, and (68) gives us,

(l) $$\int \tan.^{-1}x.dx = x\tan.^{-1}x - \tfrac{1}{2}\log.(1 + x^2) + C.$$

Ex. 4. Integrate the expressions,

$\cos.^{-1}x.dx$, $\quad$ $\text{versin}.^{-1}x.dx$, $\quad$ $\sin.^{-1}mx.dx$, $\quad$ $\tan.^{-1}mx.dx$.

By means of the formula (68), for integration by parts, we are able to integrate many transcendental quantities.

Before proceeding further with these, we will show the application of formula (68), to the reduction of differential binomials. In the form,

(69) $$x^{m-1}dx\,(a + bx^n)^p. \qquad \text{Put}$$

(m) $$dv = x^{m-1}dx, \qquad \text{and} \qquad u = (a + bx^n)^p.$$

Integrate the first of (m), differentiate the second, and substitute into the formula (68), and we have, [writing for brevity z for $a + bx^n$],

(70) $$\int x^{m-1}dx\, z^p = \frac{x^m}{m} z^p - \frac{pnb}{m} \int x^{m+n-1}dx\, z^{p-1}.$$

Here the integral of (69) depends upon another with different exponents. This is one formula.

The binomial (69) is obviously the same as

(71) $$x^{m-n}x^{n-1}dx\,(a + bx^n)^p.$$

Put $dv = (a + bx^n)^p x^{n-1}dx$, and $u = x^{m-n}$. Integrate the first, differentiate the second, and substituting into (68), we have,

(72) $$\int x^{m-1}dx\, z^p = \frac{x^{m-n}z^p}{nb\,(p+1)} - \frac{m-n}{nb\,(p+1)} \int x^{m-n-1}dx\, z^{p+1}.$$

Here the integral of (69) depends upon another with different exponents. This is another formula.

Since $(a + bx^n)^p = (a + bx^n)^{p-1}(a + bx^n) = az^{p-1} + bx^nz^{p-1}$ (69) becomes,

(73) $$\int x^{m-1}dx\, z^p = a\int x^{m-1}dx\, z^{p-1} + b\int x^{m+n-1}dx\, z^{p-1},$$

which is another formula, where the integral of (69) depends upon two other integrals.

Equate (73) and (70), and we get

(74) $$\int x^{m+n-1}dx\, z^{p-1} = \frac{x^m\, z^p}{b\,(m+pn)} - \frac{ma}{b\,(m+pn)} \int x^{m-1}dx\, z^{p-1}.$$

If in (74) we put p for $p - 1$, and m for $m + n$, it becomes,

(75) $$\int x^{m-1}\, z^p\, dx = \frac{x^{m-n}\, z^{p+1}}{b\,(m+pn)} - \frac{a\,(m-n)}{b\,(m+pn)} \int x^{m-n-1}dx\, z^p,$$

which is another formula.

If we substitute for the last term of (73), its value in (74), we get,

(76) $$\int x^{m-1}z^p dx = \frac{x^m z^p}{m+pn} + \frac{pna}{m+pn}\int x^{m-1}dxz^{p-1},$$

another formula.

If in (74) we put p for $p-1$, and solve it for the last term, we get,

(77) $$\int x^{m-1}dx\, z^p = \frac{x^m z^{p+1}}{ma} - \frac{b\,(m+n+pn)}{ma}\int x^{m+n-1}dxz^p,$$

another formula, which is useful when $m-1$ is negative.

Solve (76) for the last term, put p for $p-1$, and we have,

(78) $$\int x^{m-1}dx\, z^p = -\frac{x^m z^{p+1}}{an\,(p+1)} + \frac{m+n+pn}{an\,(p+1)}\int x^{m-1}dxz^{p+1},$$

another formula, which may be used when p is negative.

By the application of one or other of these formulas, the integral of the differential binomial (69) may generally be reduced to depend upon an integral that may be immediately obtained. As an example of the application, take the following expression,

(a) $$(a + b^2x^2)^{\frac{1}{2}}\, dx.$$

Here it may be desirable to diminish the exponent of the parenthesis. For this purpose, compare (a) with formula (76), where $b = b^2$, $m - 1 = o$, or $m = 1$, $n = 2$, $p = \frac{1}{2}$, and we have,

(b) $$\int (a+b^2x^2)^{\frac{1}{2}}dx = \frac{x}{2}(a+b^2x^2)^{\frac{1}{2}} + \frac{a}{2}\int\frac{dx}{(a+b^2x^2)^{\frac{1}{2}}}.$$

The last term of (b) is integrated at (63). Supplying the integral from (63), we have, for the integral of (a),

(79)
$$\int (a + b^2x^2)^{\frac{1}{2}}dx = \frac{x}{2}(a+b^2x^2)^{\frac{1}{2}} - \frac{a}{2b}\log.(\sqrt{a+b^2x^2} - bx) + C.$$

As another example, take the following expression,

(c) $$(R^2 - x^2)^{\frac{1}{2}}dx.$$

Apply to this the formula (76), where $a = R^2$, $b = -1$, $n = 2$, $m - 1 = o$, or $m = 1$, $p = \frac{1}{2}$, and we have,

(d) $$\int (R^2 - x^2)^{\frac{1}{2}}dx = \frac{x}{2}(R^2 - x^2)^{\frac{1}{2}} + \frac{R^2}{2}\int\frac{dx}{(R^2-x^2)^{\frac{1}{2}}}.$$

The last term of (d) is integrated at (21), viz:

$$(e) \qquad \int \frac{dx}{(R^2 - x^2)^{\frac{1}{2}}} = \sin.^{-1} \frac{x}{R} + C.$$

This put into (d), we have,

$$(80) \qquad \int (R^2 - x^2)^{\frac{1}{2}} dx = \frac{x}{2} (R^2 - x^2)^{\frac{1}{2}} + \frac{R^2}{2} . \sin.^{-1} \frac{x}{R} + C.$$

As another example, take the expression,

$$(f) \qquad \frac{x^2 dx}{(2rx - x^2)^{\frac{1}{2}}}.$$

This may be put in the form $x^{\frac{3}{2}} (2r - x)^{-\frac{1}{2}} dx$.

As it would evidently serve no purpose, in such an example, to vary the exponent of the parenthesis by unity, let us diminish the exponent of the variable without the parenthesis. For this purpose, apply formula (75) where $a = 2r$, $b = -1$, $n = 1$, $p = -\frac{1}{2}$, and $m - 1 = \frac{3}{2}$, or $m = \frac{5}{2}$, and (75) becomes

$$(g) \qquad \int x^{\frac{3}{2}} dx \, (2r - x)^{-\frac{1}{2}} = -\frac{x^{\frac{3}{2}}}{2} (2r - x)^{\frac{1}{2}} + \frac{3r}{2} \int x^{\frac{1}{2}} dx \, (2r - x)^{-\frac{1}{2}}.$$

Apply (75) again to the last term of (g), where $m - 1 = \frac{1}{2}$, or $m = \frac{3}{2}$, and the other values remain as before, and (75) becomes,

$$(h) \qquad \int x^{\frac{1}{2}} dx \, (2r - x)^{-\frac{1}{2}} = -x^{\frac{1}{2}} (2r - x)^{\frac{1}{2}} + r \int \frac{dx}{(2rx - x^2)^{\frac{1}{2}}}.$$

The last term of (h) is integrated at (34), viz:

$$(k) \qquad \int \frac{dx}{(2rx - x^2)^{\frac{1}{2}}} = \text{versin.}^{-1} \frac{x}{r}.$$

Put the value of (k) into (h), and then the value of (h) into (g), and we have,

$$(81) \qquad \left\{ \begin{aligned} \int \frac{x^2 dx}{(2rx - x^2)^{\frac{1}{2}}} &= -\frac{x}{2} (2rx - x^2)^{\frac{1}{2}} + \frac{3r}{2} \Big(-(2rx - x^2)^{\frac{1}{2}} \\ &\quad + r \text{ versin.}^{-1} \frac{x}{r} \Big) + C. \end{aligned} \right.$$

An extensive class of differentials may be integrated by the application of the formulas (70) to (78). The same could be integrated by the direct application of the formula (68).

CHAPTER VI.

INTEGRATION OF TRANSCENDENTAL FUNCTIONS.

We have already seen, from (68a) to (68e), how the integration of logarithmic functions may be effected by integrating by parts. We have also shown the application of the same formula to the integration of inverse circular functions, (68f) to (68h). It remains to show how the same process may be applied to trigonometrical lines.

Since by Differential Calculus, $d.\sin.x = \cos.xdx$, we have,

(82) $$\therefore \int \cos.xdx = \sin.x,$$

and since $d.\sin.^m x = m \sin.^{m-1}x \cos.xdx$, we have,

(83) $$\therefore \int \sin.^m x.\cos.xdx = \frac{\sin.^{m+1}x}{m+1} + C.$$

In like manner, we get for the integral of $\cos.^n x \sin.xdx$,

(84) $$\int \cos.^n x.\sin.xdx = -\frac{\cos.^{n+1}x}{n+1} + C.$$

If we had to integrate the form,

(85) $$\sin.^m x \cos.^n xdx,$$

we may assume

$$\sin.^{m-1}x = u, \quad \text{and} \quad \cos.^n x \sin.xdx = dv.$$

Differentiate the first, integrate the second, and substituting into formula (68), we have,

(86) $$\left\{ \begin{array}{l} \int \sin.^m x \cos.^n xdx = -\dfrac{\sin.^{m-1}x \cos.^{n+1}x}{n+1} + \\ \dfrac{m-1}{n+1} \int \sin.^{m-2}x \cos.^{n+2}xdx. \end{array} \right.$$

Since by Trigonometry we have,

$$\cos.^{n+2}x = \cos.^n x (1 - \sin.^2 x),$$

this value put into (86), it becomes,

(87) $$\left\{ \begin{array}{l} \int \sin.^m x \cos.^n xdx = -\dfrac{\sin.^{m-1}x \cos.^{n+1}x}{m+n} + \\ \dfrac{m-1}{m+n} \int \sin.^{m-2}x \cos.^n xdx. \end{array} \right.$$

Here the integral is made to depend upon another where the ex-

ponent of $\sin.^m x$ is diminished by 2. By the successive application of this formula, we can, when m is odd, reduce any differential of the proposed form to the form (84), and obtain its complete integral.

Ex.—Integrate the expression,

$$(a) \qquad \sin.^3 x \cos.^2 x dx.$$

Comparing this to (87), we have, $m = 3$, $n = 2$, and (87) becomes,

$$(b) \quad \therefore \quad \int \sin.^3 x \ \cos.^2 x dx = -\frac{\sin.^2 x \cos.^3 x}{5} + \frac{2}{5}\int \cos.^2 x . \sin. x dx.$$

Integrate the last term of (b), as in (84), and substituting into (b), we have, for the complete integral of (a),

$$(c) \quad \int \sin.^3 x \cos.^2 x dx = -\frac{\cos.^3 x}{5}\left(\sin.^2 x + \frac{2}{3}\right) + \text{C}.$$

Ex. 2.—Integrate $\sin.^5 x \cos.^4 x dx$.

If m be even and n odd, we could, by taking as parts of (85),

$$\cos.^{n-1} x = u, \quad \text{and} \quad \sin.^m x \cos. x dx = dv,$$

and proceeding as before, obtain the formula,

$$(88) \quad \left\{ \begin{array}{l} \displaystyle\int \sin.^m x \cos.^n x dx = \frac{\cos.^{n-1} x \sin.^{m+1} x}{m+n} + \\ \displaystyle\frac{n-1}{m+n}\int \cos.^{n-2} x \ \sin.^m x dx. \end{array} \right.$$

When n is odd, (88) reduces the form (85) to depend on (83).

If n be zero, (85) becomes $\sin.^m x dx$, and the form for integrating is found by putting n zero in (87) which gives,

$$(89) \quad \int \sin.^m x dx = -\frac{\sin.^{m-1} x \cos. x}{m} + \frac{m-1}{m}\int \sin.^{m-2} x dx.$$

By the successive application of this formula, the integral will, when m is even, finally depend upon $\int dx$, and when m is odd, upon $\int \sin. x dx$, and thus the complete integral be obtained.

If m be zero, (88) becomes,

$$(90) \quad \int \cos.^n x dx = \frac{\cos.^{n-1} x \ \sin. x}{n} + \frac{n-1}{n}\int \cos.^{n-2} x dx.$$

By the successive application of this formula, the integral will, when n is even, finally depend upon $\int dx$, and when n is odd, upon $\int \cos. x dx$.

Ex.—Integrate $\cos.^4 x dx$.

Employ formula (90) where $n = 4$.

$$(d) \qquad \int \cos.^4 x dx = \frac{\cos.^3 x \sin.x}{4} + \tfrac{3}{4} \int \cos.^2 x dx.$$

Apply the same formula to the last term of (d), where $n = 2$,

$$(e) \quad \int \cos.^2 x dx = \frac{\cos.x \sin.x}{2} + \tfrac{1}{2} \int dx = \frac{\cos.x \sin.x}{2} + \tfrac{1}{2}\ x.$$

Put this value of (e) into (d), and we have the entire integral.

Ex.—Integrate $\cos.^3 x dx$, $\sin.^4 x dx$, $\sin.^3 x dx$.

If m be negative, the form (85) can be reduced by formula (88), to depend upon the integral of $\sin.^{-m} x \cos.x dx$, when n is odd, and upon $\sin.^{-m} x dx$, when n is even. The first of these is immediately integrable by (83).

It remains to integrate the form,

$$(91) \qquad \sin.^{-m} x dx.$$

Solve (89) for the last term of the second side, and putting m for $m - 2$, we have,

$$(92) \quad \int \sin.^m x dx = \frac{\sin.^{m+1} x \ \cos.x}{m+1} + \frac{m+2}{m+1} \int \sin^{m+2} x dx.$$

If m be negative, formula (92) will serve to reduce (91) to depend upon $\int dx$, when m is even, and upon

$$(93) \qquad \int \frac{dx}{\sin.x},$$

when m is odd.

To integrate the form (93), multiply its numerator and denominator by $\sin.x$, and it may be put in the form,

$$(94) \qquad \int \frac{\sin.x dx}{1 - \cos.^2 x}.$$

Put $z = \cos.x \quad \therefore \quad -dz = \sin.x dx$, and $z^2 = \cos.^2 x$.

These values put into (94), it becomes,

$$(95) \qquad \int \frac{-dz}{1 - z^2}.$$

This integrated as a rational fraction, we have,

$$(96) \qquad \int \frac{-dz}{1 - z^2} = \tfrac{1}{2} \log.\left(\frac{1 - z}{1 + z}\right) + C,$$

or restoring the value of z,

(97) $$\int \frac{dx}{\sin.x} = \tfrac{1}{2}\log.\left(\frac{1-\cos.x}{1+\cos.x}\right) + C.$$

In a similar manner, if n were negative, (85) could, by formula (87), be reduced to depend upon the integral of $\cos.^{-n}x \sin.xdx$, when m is odd, and upon $\cos.^{-n}xdx$, when m is even. The first of these is immediately integrable by (84). It remains to integrate the form,

(98) $$\cos.^{-n}xdx.$$

Solve (90) for the last term of the second side, and putting n for $n-2$, we get,

(99) $$\int \cos.^{n}xdx = -\frac{\cos.^{n+1}x}{n+1} + \frac{n+2}{n+1}\int \cos.^{n+2}xdx.$$

By the application of this formula, (98) may be reduced to depend upon $\int dx$ when n is even, or upon

(100) $$\int \frac{dx}{\cos.x},$$

when n is odd.

To integrate (100), multiply numerator and denominator by $\cos.x$, and proceeding as with (93), we get,

(101) $$\int \frac{dx}{\cos.x} = \tfrac{1}{2}\log.\left(\frac{1+\sin.x}{1-\sin.x}\right) + C.$$

Ex.—Integrate

$$\frac{\sin.^2xdx}{\cos.^4x}, \quad \frac{\cos.^2xdx}{\sin.^3x}, \quad \frac{dx}{\cos._3x}, \quad \frac{dx}{\sin.^4x}.$$

The preceding deductions of formulæ show the method of integrating by parts. By means of the formula now deduced, many binomial and trigonometrical functions can be integrated. A very common practice, however, is to take the formula (68) for integrating by parts, and proceed, with its aid, to reduce the integral to some form readily integrable. In this way, transcendental functions may, in many complicated cases, be integrated.

Transcendental functions may, in many cases, be changed to the form of algebraic functions, and treated by the methods already presented. Thus to integrate

$$(102) \qquad \frac{dx}{a + b \cos.x},$$

we may put $\cos.x = z$, $\therefore$ $x = \cos.^{-1} z$, and (102) is, by substitution, changed to the form,

$$(103) \qquad \frac{-\,dz}{(a + bz)\,(1 - z^2)^{\frac{1}{2}}},$$

which may be rationalised by the method of irrational fractions. Or if we assume

$$(104) \qquad \cos.x = \frac{1 - z^2}{1 + z^2},$$

and substitute into (102), it becomes,

$$(105) \qquad \frac{2dz}{a + b + (a - b)z^2},$$

which may be integrated by rational fractions, if b be greater than a, or by a circular arc, if b be less than a.

But we will not pursue this subject further. The student who desires ample details of these matters, is referred to La Croix's Calculus, volume II., where all the valuable labours of Analysts on the subject are collected.

CHAPTER VII.

INTEGRATION BY SERIES.

When none of the preceding methods enable us to integrate a proposed differential of one variable, the integral may be approximated to by a series.

The Theory of Integration by Series may be readily understood. For let the expression to be integrated be represented by $\varphi x.dx$. Let this be divided into two parts, as $\varphi' x$, and $\varphi'' xdx$, and let the part $\varphi' x$ be developed into a series by any of the methods of development applicable to it, [as by division, or by the Binomial Theorem, or by M'Laurin's Theorem, &c.], then multiply each term of this development by the other part, $\varphi'' xdx$, and each term being then integrated separately, we have the integral of the proposed $\varphi x.dx$.

As an example, let the expression to be integrated be,

$$(a)\qquad \frac{dx}{1+x}.$$

To integrate this by series, divide it into the two parts $\frac{1}{1+x}$ and dx. The first part being developed by division, is

$$(b)\qquad \frac{1}{1+x} = 1 - x + x^2 - x^3 + \&c.$$

Multiply this by dx, the second part, and we have,

$$(c)\qquad \therefore \frac{dx}{1+x} = dx - xdx + x^2dx - x^3dx.$$

Integrating this, we have,

$$(d)\qquad \int \frac{dx}{1+x} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c. + C.$$

This second side of (d) is an approximation to the integral.

Ex. 2. Integrate the expression,

$$(105e)\qquad \frac{(1-e^2x^2)^{\frac{1}{2}}dx.}{(1-x^2)^{\frac{1}{2}}}.$$

This may be divided into the two parts,

$$(1-e^2x)^{\frac{1}{2}}, \qquad \text{and} \qquad \frac{dx}{(1-x^2)^{\frac{1}{2}}}.$$

The first part developed into a series by the Binomial Theorem, is,

$$(f)\qquad (1-e^2x^2)^{\frac{1}{2}} = 1 - \frac{e^2x^2}{z} + \frac{e^4x^4}{2.4} - \frac{3e^6x^6}{2.4.6} + \&c.$$

Multiply this series by the second part, $\frac{dx}{(1-x^2)^{\frac{1}{2}}}$, and the second side becomes a series, of which each term may be integrated by the preceding methods.

Ex. 3d. Integrate the expression,

$$(g)\qquad (R^2 - x^2)^{\frac{1}{2}}dx.$$

This may be divided into the two parts, $(R^2 - x^2)^{\frac{1}{2}}$, and dx. Develope the first by the Binomial Theorem, and we have,

$$(h)\qquad \therefore (R^2 - x^2)^{\frac{1}{2}} = R - \frac{x^2}{2R} - \frac{x^4}{2.4\ R^3} - \&c.$$

Multiply this by dx, the second part, and integrate, and we have,

$$(k) \qquad \int (\mathrm{R}^2 - x^2)^{\frac{1}{2}} dx = \mathrm{R}x - \frac{x^3}{2.3\,\mathrm{R}} - \frac{x^5}{2.4.5\,\mathrm{R}^3} - \&c. + \mathrm{C}.$$

Ex. 4. Integrate by series, the expressions,

$$(x^2 - 1)^{-\frac{1}{2}} dx, \qquad \frac{dx}{x^n + a}, \qquad \frac{dx}{(1 + x^2)^{\frac{1}{2}}}, \qquad \frac{dx}{(1 - x^2)^{\frac{1}{2}}}.$$

CHAPTER VIII.

INTEGRATION OF DIFFERENTIALS EXCEEDING THE FIRST ORDER.

The second differential coefficient, third differential coefficient, &c., are called differential coefficients of the higher orders. Suppose we have,

$$(106) \qquad \frac{d^3y}{dx^3} = mx.$$

Multiply by dx, and we have,

$$(107) \qquad \frac{d^3y}{dx^2} = mxdx.$$

Integrate and we have,

$$(108) \qquad \frac{d^2y}{dx^2} = \frac{mx^2}{2} + \mathrm{C}.$$

Multiply again by dx, and integrate, and we have,

$$(109) \qquad \frac{dy}{dx} = \frac{mx^3}{2.3} + \mathrm{C}x + \mathrm{C}'.$$

Multiply again by dx, and integrate, and we have,

$$(110) \qquad y = \frac{mx^4}{2.3.4} + \frac{\mathrm{C}x^2}{2} + \mathrm{C}'\,x + \mathrm{C}''.$$

This is the complete integral of (106), a constant being added at each integration. This example shows the process of integrating in such cases.

Integrate $\dfrac{d^4y}{dx^4} = mx^2$, and $\dfrac{d^5y}{dx^5} = n$.

An integral of the second order is denoted by the form $\int^2$, or by

repeating the sign of integration, thus $\int\int$. In the same way, the third integral of $axdx^3$, may be denoted by

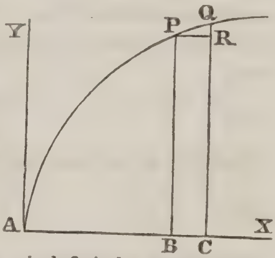
$$a\int^3 xdx^3, \qquad \text{or by} \qquad a\int\int\int xdx^3.$$

We shall now give a number of Geometrical Applications of the Integral Calculus, which we will put into the form of Propositions, for the sake of convenience.

PROPOSITION I.

Determine the area of a plane curve.

FIG. 77.



Let AP be the curve, whose equation is

(111) $$y = \varphi x.$$

Let (x,y) be the point P on the curve. Then dx may be represented by BC. Let the area be reckoned between the curve and the axis of X. If ABP be the area, BCQP would represent the differential of the area, being an indefinitely small increment of it. Now, by the figure, we have, ydx = area of the rectangle BR. When BC or dx is indefinitely small, the rectangle BR becomes the differential area BCQP, that is, putting A for the area of ABP

(112) $$d\text{A} = ydx,$$

and integrating this,

(113) $$\text{A} = \int ydx.$$

Such is the general form for the area contained between the curve and the axis AX.

The integration of (112) was regarded, by the early writers on the Calculus, as the addition of the elemental areas BR, or rather BCQP.

To apply (113) to any given curve (111), we eliminate one of the co-ordinates in (113) by means of the equation of the curve. Thus, substitute from (111) into (113) the value of y, and we have,

(114) $$A = \int \phi x . dx,$$

which being a function of a single variable, may be integrated. [See Problem F, post.]

Ex.—Find the area of the common parabola.

Here (111) becomes,

(*a*) $$y = p^{\frac{1}{2}} x^{\frac{1}{2}}.$$

This value of y put into (113), we have,

(*b*) $$A = p^{\frac{1}{2}} \int x^{\frac{1}{2}} dx = \frac{2}{3} p^{\frac{1}{2}} x^{\frac{3}{2}} + C,$$

the area required.

Before proceeding further with examples under this Proposition, we will solve the following Proposition.

PROPOSITION II.

Determine the constant added in integration.

To explain the method of doing this, take the following example under the preceding Proposition. Find the area of the curve,

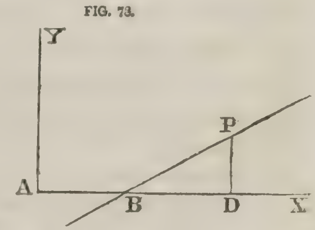
(*c*) $$y = ax - b.$$

Put this value of y into (113), and integrate, and we have,

(*d*) $$A = \frac{ax^2}{2} - bx + C.$$

This is the general expression for the area.

FIG. 73.



If now we wish the area PBD, bounded on the abscissa by BD, we must obviously subject the integral (*d*) to such conditions as will express this area.

Let AB $= m$, and AD $= n$. If we estimate the area from B towards PD, it is obvious that when $x = m =$ AB, the area (*d*) is zero. Put then in (*d*), $x = m$, and it becomes,

(*e*) $$o = \frac{am^2}{2} - bm + C.$$

This equation is sufficient to determine the constant C, and the value of it from (e) put into (d), would give us the area of PBD for any abscissa x. If a definite value n be given to AD, then if $x = n$, (d) becomes,

$$(f) \qquad \text{Area} = \frac{an^2}{2} - bn + C.$$

If (e) be subtracted from (f), we have,

$$(g) \qquad \text{Area} = \frac{a}{2}(n^2 - m^2) - b(n - m).$$

In this we have the area required, independently of any indeterminate constant.

If (e) be subtracted from (d), we have,

$$(h) \qquad \text{Area} = \frac{a}{2}(x^2 - m^2) - b(x - m),$$

where the constant is also eliminated, and the base BD of the triangle becomes fixed by giving to x any value greater than m.

Equation (d) may be regarded as containing three unknown quantities, viz: x, C, and A. The conditions to which we subject the particular proposition, enables us to determine two of them, viz: x and A, and gives (e) an equation which makes known the value of the indeterminate C.

This is called *Integration between Limits*. The limits that give the determinate area (g) are $x = m$, and $x = n$. This is usually denoted by writing the limits in the form of a fraction after the sign of integration, thus $\int \frac{n}{m}$, the subtractive limit being in the denominator. Thus the expression, $\int \frac{n}{o} (ax - x^2)\, dx$, means that the constant after integration is to be eliminated by taking the integral between the limits $x = o$, and $x = n$. Equation (h) is made by taking the integral between the limits $x = m$, and $x = x$.

The constant may be zero, but *Integration between Limits* is the process substantially employed, in all cases, to determine the constant and define the integral. The object aimed at in each particular proposition, must designate the limits, between which the integral is to be taken.

We will add some examples illustrative of these two propositions.

Ex. 1. Determine the area of a common parabola between the limits $x = o$, and $x = x$.

This is done by taking equation (b), Proposition I. between these limits. For $x = o$, the area is zero, and (b) becomes, $o = C$. Hence (b) is simply

$$(k) \qquad A = \frac{2}{3} p^{\frac{1}{2}} x^{\frac{3}{2}} = \frac{2}{3} xy,$$

by eliminating p by means of equation (a). Hence the area of the common parabola reckoned from the vertex is two-thirds of the circumscribing rectangle.

Ex. 2. Determine the area of the circle.

$$(l) \qquad y = (R^2 - x^2)^{\frac{1}{2}},$$

is the equation of the circle. This value of y put into (113), we have,

$$(m) \qquad A = \int (R^2 - x^2)^{\frac{1}{2}} dx,$$

an equation integrated at (80), where we have,

$$(n) \qquad \therefore \quad A = \frac{x}{2} (R^2 - x^2)^{\frac{1}{2}} + \frac{R^2}{2} \sin.^{-1} \frac{x}{R} + C.$$

This between the limits $x = o$, and $x = R$, gives the area of a quadrant of the circle. Estimating the area from the axis of Y, when $x = o$, we have, $A = o$, and $C = o$. Hence C may be omitted in (n). For the other limit, if $x = R$, (n) becomes,

$$(o) \qquad A = \frac{R^2}{2} \sin.^{-1} 1.$$

But $\sin.^{-1} 1 = 90° = \frac{\pi}{2}$. Hence (o) is

$$(p) \qquad A = \frac{R^2}{2} . \frac{\pi}{2} = \text{quadrant};$$

and four times this is,

$$(q) \qquad \text{circle} = \pi R^2,$$

which agrees with the result in Plane Geometry.

Ex. 3.—Determine the area of the ellipse.

The equation of the ellipse solved for y, is,

$$(r) \qquad y = \frac{b}{a} (a^2 - x^2)^{\frac{1}{2}}.$$

This value of y put into (113), we have,

$$(s) \qquad A = \frac{b}{a} \int (a^2 - x^2)^{\frac{1}{2}} dx.$$

The part under the sign of integration is by (m) the area of a circle whose radius is a. Hence if a circle be described on the major diameter of an ellipse, the area of this circle multiplied by the minor diameter, and divided by the major diameter, is the area of the ellipse.

Ex. 4. Determine the area of the cycloid.

The differential equation of the cycloid is, [Appendix (22), Differential Calculus],

$$(2a) \qquad dx = \frac{ydy}{(2ry - y^2)^{\frac{1}{2}}}.$$

This value of dx put into (113), we have,

$$(2b) \qquad A = \int \frac{y^2dy}{(2ry - y^2)^{\frac{1}{2}}}.$$

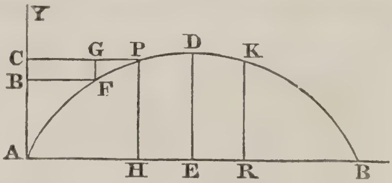
This is integrated at (81), from which we have,

(2c)

$$A = -\frac{y}{2}(2ry - y^2)^{\frac{1}{2}} + \frac{3r}{2}\left(-(2ry - y^2)^{\frac{1}{2}} + r \text{ versin.}^{-1}\frac{y}{r}\right) + C.$$

Between the limits $y = o$, $y = 2r$, this gives us the area of ADE, half of the cycloid. When $y = o$, $A = o$, and $C = o$.

FIG. 79.



When $y = 2r$, (2c) becomes,

$$(2d) \qquad A = \frac{3r^2}{2} \text{ versin.}^{-1}2.$$

But

$$\text{versin.}^{-1}2 = 180° = \pi.$$

This put into (2d), and that equation doubled, we get

$$(2e) \qquad \text{area of cycloid} = 3\pi r^2,$$

which shows the area of the cycloid to be three times the area of the generating circle.

Ex.—Determine the area of the following curves :—

$y = ax^n$, $\quad y = a^x$, $\quad y = a \sin.x$, $\quad y = a \sin.^4x \cos.^3x$.

This last value of y put into (113), we have,

(2f) $$A = a\int \sin.^4 x \cos.^3 x dx,$$
whose integral is found by (88).

Cor. 1.—If we take, fig. 79, BC $= dy$, and BF $= x$, we would find, instead of (113),

(115) $$d.\,A = xdy,$$
where the area would be the part between the axis of y and the curve. If the equation of the curve be

(116) $$x = \psi y,$$
this value of x put into (115), and that equation integrated, we have,

(117) $$A = \int \psi y.dy.$$

Cor. 2.—In determining the differential area (112), the axes were supposed rectangular, or BR, fig. 77, was a rectangle. If the axes be inclined at an angle φ, it is obvious that ydx must be multiplied by $\sin.\varphi$, to express the differential of the area. In this case (113) becomes,

(118) $$A = \sin.\varphi \int ydx.$$

As an example of the application of (118), find the area of the curve,

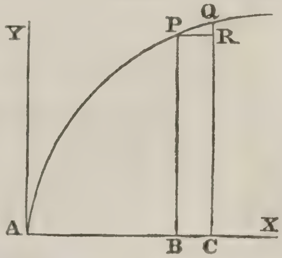
(2g) $$xy = m^2,$$
which is the hyperbola referred to its asymptotes. Let φ be the angle of inclination of the asymptotes. The value of y from (2g), put into (118), and that equation integrated, we have,

(2h) $$A = m^2 \sin.\varphi \log.x + C,$$
which may be taken between any proposed limits.

PROPOSITION III.

Determine the length of a given curve.

FIG. 80.



Let (x,y) be the co-ordinates of P, Fig. 80.

Let Q be indefinitely near to P, and we have, $dx =$ PR, $dy =$ QR.

Let AP $= z$, $\therefore$ PQ $= dz$.

Suppose the curve a polygon of an indefinite number of sides, one of which is PQ, then from the triangle PQR, [the axes being rectangular], we have,

(119) $dz^2 = dx^2 + dy^2 \quad \therefore \quad dz = (dx^2 + dy^2)$.

The integral of this gives the arc z, viz:

(120) $$z = \int (dx^2 + dy^2)^{\frac{1}{2}}.$$

If the equation of the curve be

(121) $$y = \phi x,$$

we can, by means of this equation, express the second side of (120) in terms of one variable, and integrate it. We may put (120) into the form,

(122) $$z = \int (1 + p^2)^{\frac{1}{2}} dx, \quad \text{where} \quad p = \frac{dy}{dx}, \quad \text{or,}$$

(123) $$z = \int (1 + q^2)^{\frac{1}{2}} dy, \quad \text{where} \quad q = \frac{dx}{dy},$$

which are convenient in practice.

Ex. 1. Determine the length of an arc of the parabola.

(a) $$y^2 = 4mx,$$

is the equation of the curve. Differentiate this, and we have,

(b) $$\frac{dx}{dy} = \frac{y}{2m} = q.$$

This put into (123), we have,

(c) $$z = \frac{1}{2m} \int (4m^2 + y^2)^{\frac{1}{2}} dy.$$

This is integrated at (79), by making $a = 4m^2$, and $b = 1$, from which we have

(d) $$z = \frac{y}{4m} (4m^2 + y^2)^{\frac{1}{2}} - m \log.(\sqrt{4m^2 + y^2} - y) + \text{C}.$$

This taken between the limits $y = o$, to $y = y$, will give the length of the curve for any ordinate y. For $y = o$, the arc $z = o$, and (d) becomes,

$$o = -m \log.2m + \text{C}, \quad \therefore \quad \text{C} = m \log.2m,$$

which may be put into (d).

Ex. 2. Determine the length of the cycloid.

From the differential equation of the cycloid ($2a$), last Proposition, we get,

(f) $$\frac{dx^2}{dy^2} = \frac{y}{2r - y} = q^2.$$

This value put into (123), we get,

$$(g) \qquad z = \sqrt{2r} \int (2r - y)^{-\frac{1}{2}} dy.$$

This integrated by Rule V., we have,

$$(h) \qquad z = -2\sqrt{2r}\,(2r - y)^{\frac{1}{2}} + C.$$

For half of the cycloid AD, [fig. 79], take (h) between the limits $y = o$, $y = 2r$, when $y = o$, z is zero, and (h) becomes,

$$(k) \qquad o = -4r + C.$$

When $y = 2r$, (h) becomes,

$$(l) \qquad z = C.$$

Subtract (k) from (l), and we have,

$$(m) \qquad z = 4r.$$

The double of this, viz: $8r$, is the length of the whole arc of the cycloid.

Subtract (k) from (h), and we have,

$$(n) \qquad \therefore \quad z = 4r - 2\sqrt{2r}\,(2r - y)^{\frac{1}{2}},$$

which is the length of an arc DP, measured from the vertex D to any point P whose ordinate is y.

Ex. 3. Determine the length of the ellipse.

Take as the equation of the ellipse,

$$(o) \qquad y^2 = (e^2 - 1)(x^2 - a^2).$$

The value of dy deduced from (o), in terms of x, and put into (120), gives, after putting $x = ax'$,

$$(p) \qquad z = a \int \frac{(1 - e^2x'^2)^{\frac{1}{2}} dx'}{(1 - x'^2)^{\frac{1}{2}}},$$

which may be integrated in a series, as directed at (105e).

Ex. 4. Determine the length of the semicubical parabola.

$y^3 = ax^2$ is the equation. Let the limits be $x = o$, $x = m$.

Ex. 5. Find the length of the curve,

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = m^{\frac{2}{3}}, \text{ between the limits } x = o,\ x = m.$$

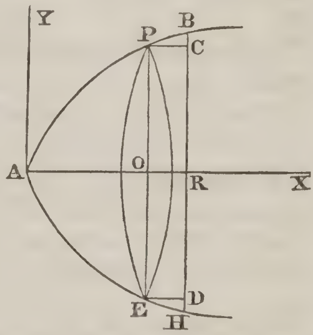
Ans. $\quad z = \dfrac{3}{2} m.$

PROPOSITION IV.

Determine the area of a surface of revolution.

If the curve AB revolve round the fixed axis AX, it generates a surface of revolution. For the purpose of applying the Integral Calculus, the surface of revolution may be regarded as generated by the perimeter of a circle which moves with its centre on the axis of revolution AX, has its plane perpendicular to that axis, and the curve AB in the plane of the paper for its directrix.

FIG. 81.



Let the equation of the directrix be,

(124) $y = \varphi x,$

and let (x,y) be the point P. Since y is the radius of the generating circle, $2\pi y$ is the perimeter of the generating circle.

Suppose the points P and B indefinitely near to each other, then the half sum of the perimeters of the circles whose diameters are PE and BH multiplied by PB, gives the area of the conic frustum generated by the revolution of PB round AX. The area of this conic frustum is the elementary increment of the surface or differential of the surface. But since P and B are indefinitely near, the perimeter of PE may be taken as the half sum of the perimeters of the circles PE and BH. Hence calling the surface of revolution S, we have,

(125) $d\text{S} = 2\pi y \times \text{PB}.$

Substitute in this for PB, its value dz in (119), we have,

(126) $d\text{S} = 2\pi y.\ (dx^2 + dy^2)^{\frac{1}{2}},$

and integrating this, we have,

(127) $\text{S} = 2\pi \int y\ (dx^2 + dy^2)^{\frac{1}{2}}.$

By means of (124), the equation of the directrix, the second side of (127) may be expressed in terms of a single variable.

Ex. 1. Determine the surface of the sphere.

(*a*) $$y = (2rx - x^2)^{\frac{1}{2}},$$

is the equation of the directrix AB in this case. From (*a*) we get,

(*b*) $$dy^2 = \frac{(r-x)^2}{2rx - x^2} \cdot dx^2.$$

This value, and that of y in (*a*), put into (127), we have,

(*c*) $$S = 2\pi r \int dx = 2\pi rx + C.$$

This between the limits $x = o$, $x = 2r$, gives for the surface of the sphere,

(*d*) $$S = 4\pi r^2 = \text{four great circles.}$$

Ex. 2. Determine the surface generated by the revolution of a cycloid about its base.

The differential equation of the cycloid is, [Appendix, Differential Calculus, (22),]

(*e*) $$dx = \frac{ydy}{(2ry - y^2)^{\frac{1}{2}}}$$

This value of dx put into (127), we have,

(*f*) $$S = 2\pi \sqrt{2r} \int (2r - y)^{-\frac{1}{2}} ydy.$$

This integrated by formula (75) is,

(*g*) $$S = -\frac{4}{3}\pi \sqrt{2ry}\,(2r - y)^{\frac{1}{2}} - \frac{16}{3}\pi r \sqrt{2r}\,(2r - y)^{\frac{1}{2}} + C.$$

This between the limits $y = o$, $y = 2r$, gives for the area of half the surface,

(*h*) $$S = \frac{32}{3}\pi r^2,$$

and double of this is the whole surface.

Ex. 3. Determine the surface of the cone.

Here $y = ax$ is the equation of the directrix.

Ex. 4. Determine the surface of the ellipsoid of revolution.

Ex. 5. Determine the surface of the paraboloid of revolution.

Ex. 6. Determine the surface generated by the revolution of the curve, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = m^{\frac{2}{3}}$, round the axis of X.

This between the limits $x = o$, $x = m$, gives for the surface,

(*k*) $$S = 2\pi m^2 = \text{areas of two circles of radius } m.$$

Ex. 7. Determine the area generated by the revolution of a cycloid round its axis, [DC fig. 79.]

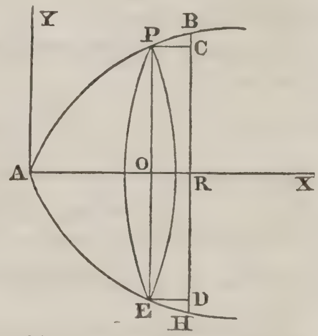
Ans. $S = 8\pi r^2 (\pi - \frac{4}{3})$.

PROPOSITION V.

Determine the volume of a solid of revolution.

The solid of revolution may be conceived to be generated by the area of a circle moving as in last Proposition.

FIG. 82



If (x,y) be the point P, and

(128) $$y = \varphi x,$$

the equation of the directrix AB, then the area of the generating circle is πy^2. This multiplied by $dx = OR$, we have, $\pi y^2 dx$, which is the volume of the cylinder whose base is the circle PE, and whose altitude is $OR = dx$.

When the points P and B are indefinitely near, this cylinder becomes the same as the solid EHBP, which is the differential of the volume. That is putting V for the volume, we have,

(129) $$dV = \pi y^2 dx.$$

This integrated, gives the volume,

(130) $$V = \pi \int y^2 dx.$$

By means of (128), this may be expressed in terms of one variable, and integrated.

Ex. 1. Find the volume of the sphere.

(*a*) $$y^2 = 2rx - x^2,$$

is the equation of the directrix. This put into (130) and integrated, we have,

(*b*) $$V = \pi \left(rx^2 - \frac{x^3}{3}\right) + C.$$

This between the limits $x = o$, $x = 2r$, gives for the volume of the sphere,

$$(c) \qquad V = \frac{4}{3}\pi r^3.$$

Ex. 2. Find the volume of the ellipsoid of revolution,

$$(d) \qquad y^2 = \frac{b^2}{a^2}(a^2 - x^2).$$

This put into (130), and integrated between the limits $x = o$, $x = a$, give the volume of half the solid.

Ex. 3. Find the volume of the paraboloid of revolution,

$$(e) \qquad y^2 = px,$$

is the directrix. This put into (130), and integrated, gives,

$$(f) \qquad V = \frac{\pi p x^2}{2},$$

which needs no correction, for C, V, and x are all zero together. Eliminate p between (e) and (f), and we have,

$$(g) \qquad V = \frac{\pi x y^2}{2} \text{ equal half the circumscribing cylinder.}$$

Ex. 4. Find the volume of the cone.

$y = ax$, is the directrix.

Ex. 5. Find the volume generated by a cycloid revolving round its base.

Cor.—If the axis of y be the axis of revolution, instead of (130) we would obviously have,

$$(131) \qquad V = \pi \int x^2 dy.$$

Ex.—Find the volume generated by the revolution of a parabola round the tangent to its vertex.

The axis of y is the axis of revolution, and the value of x from (e), put into (131), and that equation integrated, we have,

$$(h) \qquad V = \frac{\pi y x^2}{5}.$$

Ex.—Find the volume generated by the revolution of the curve, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = m^{\frac{2}{3}}$, first round the axis of y, second round the axis of x, between the limits $y = o$, $y = m$, and $x = o$, $x = m$.

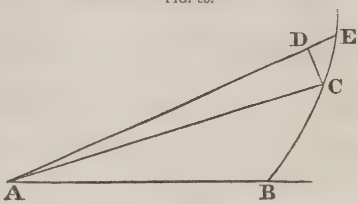
Ans. $v = \frac{16}{105} m^3.$

PROPOSITION VI.

Determine the area of a plane curve referred to polar co-ordinates.

Let BE be the curve, AB the angular axis, AC the radius vector $= \rho$, CAB $= \omega$, and CD perpendicular to AC. Then by Differential Calculus, [Proposition XXXIII., (169),] we have, when E and C are indefinitely near to each other, CD $= \rho d\omega$.

FIG. 83.



Then the triangle ACD has for its area, $\frac{\rho^2 d\omega}{2}$, which may be taken to be the same as the area ACE, the elemental area or differential of the area CAB. Hence if A be the area,

(132) $$dA = \frac{\rho^2 d\omega}{2},$$

and integrating this, we have,

(133) $$A = \tfrac{1}{2}\int \rho^2 d\omega.$$

If the equation of the curve be,

(134) $$\rho = \varphi\omega,$$

we can, by means of this, express (133) in terms of one variable, and then integrate it.

Ex. 1. Determine the area of the curve,

(*a*) $\rho = a\omega$, between the limits $\rho = m$, and $\rho = n$.

Differentiate (*a*), and we have,

(*b*) $$d\omega = \frac{d\rho}{a},$$

which put into (133), we have,

(*c*) $$A = \frac{1}{2} \, a \int \rho^2 d\rho = \frac{1}{6a}\rho^3 + C,$$

and this between the limits proposed is,

(*d*) $$A = \frac{1}{6a}(n^3 - m^3).$$

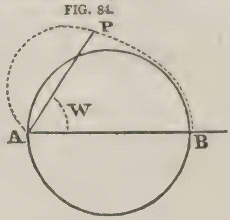
We might also have found this in terms of ω, by putting into (133) the value of ρ in (a).

Ex. 2.—Determine the area of the circloid.

$$(e) \qquad \rho = a \cos.\omega + a,$$

is the equation of the curve, [Differential Calculus, (136f),].

FIG. 84.



Let us express (133) in terms of ρ. For this purpose, differentiate (e), and we get,

$$(f) \qquad d\omega = -\frac{d\rho}{a \sin.\omega}.$$

Put into (e) for $\cos.\omega$, its equal,

$$(1 - \sin.^2\omega)^{\frac{1}{2}},$$

and solving the result for $\sin.\omega$, we get,

$$(g) \qquad a \sin.\omega = (2a\rho - \rho^2)^{\frac{1}{2}},$$

which put into (f), gives,

$$(h) \qquad d\omega = -\frac{d\rho}{(2a\rho - \rho^2)^{\frac{1}{2}}}.$$

This value put into (133), we have,

$$(k) \qquad A = -\frac{1}{2}\int \frac{\rho^2 d\rho}{(2a\rho - \rho^2)^{\frac{1}{2}}}.$$

This is integrated at (81), from which we have,

$$(l) \ A = \frac{1}{4}\rho(2a\rho - \rho^2)^{\frac{1}{2}} - \frac{a}{4}\left(-(2a\rho - \rho^2)^{\frac{1}{2}} + a \text{ versin.}^{-1}\frac{\rho}{a}\right) + C.$$

This taken between the limits $\rho = o$, and $\rho = 2a$, we have,

$$(m) \qquad A = \frac{3}{4}\pi a^2,$$

which is the area above the angular axis AB.

The double of (m) gives the whole area of the circloid, viz:

$$(n) \qquad \text{circloid} = \frac{3}{2}\pi a^2.$$

Hence the area of the circloid is three semicircles of the directing circle.

If we substitute into (133) for ρ^2, its value from (e), we have,

$$(o) \qquad A = \tfrac{1}{2}\int (a^2\cos.^2\omega d\omega + a^2 d\omega + 2a^2\cos.\omega d\omega),$$

where each term may be integrated separately, the first term under the vinculum being of the form (90). The integral of (o) between the limits, $\omega = o$, and $\omega = 180°$, will, as before, give us (m).

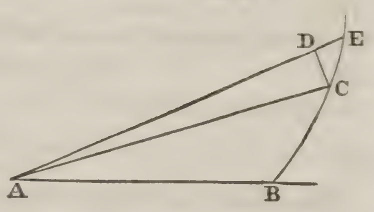
Ex. 3. Determine the area of the curves,

$$\rho = a\omega^n,\ \rho^2 = a^2\cos.2\omega, \text{ from } \omega = o \text{ to } \omega = 180°.$$

PROPOSITION VII.

Determine the length of a curve referred to polar co-ordinates.

FIG. 85.



Let BC be the curve, and z its length. Then EC represents dz, and the elemental triangle EDC gives,

(135) $EC^2 = CD^2 + DE^2$,

which since $DC = \rho d\omega$, and $DE = d\rho$, becomes,

(136) $$dz = (\rho^2 d\omega^2 + d\rho^2)^{\frac{1}{2}}.$$

Integrating (136), we have,

(137) $$z = \int (\rho^2 d\omega^2 + d\rho^2)^{\frac{1}{2}}.$$

If the curve be represented by

(138) $$\rho = \varphi\omega,$$

by means of this equation we may express the second side of (137) in terms of a single variable, and then integrate between any assigned limits.

Ex. 1. Determine the length of the curve,

(a) $$\rho = a\omega.$$

By means of (a) we may eliminate either ρ or ω from (137). To eliminate ω, differentiate (a), and substitute the value of $d\omega$ into (137), and we have,

(b) $$z = \frac{1}{a}\int (a^2 + \rho^2)^{\frac{1}{2}} d\rho,$$

which is integrated at (79).

Ex. 2. Determine the length of the circloid.

(c) $$\rho = a \cos.\omega + a,$$

is its equation. [Differential Calculus, Proposition XXIV., (136f).]

Put the value of $d\omega$ from (h), last Proposition, into (137), and we get,

(d) $$z = \sqrt{2a}\int(2a - \rho)^{-\frac{1}{2}}d\rho = -2\sqrt{2a}\,(2a - \rho)^{\frac{1}{2}} + C.$$

This between the limits $\rho = o$, and $\rho = 2a$, gives for the length of BPA, fig. 84, half of the curve,

(e) $$z = 4a,$$

and the double of this, $8a$, is the whole length, which is therefore equal to the perimeter of the square circumscribing the directing circle.

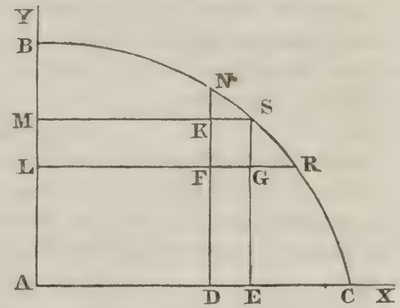
Ex. 3. Determine the length of the curve,

(f) $$\rho\omega = a.$$

PROPOSITION VIII.

Determine the area of a plane curve by double integration.

FIG. 86.



Let BC be the curve.

Let (x,y) be the point N. Let the point S be indefinitely near to N. Then $dx = \text{DE}$, $dy = \text{NK}$.

Make $\text{LM} = \text{NK} = dy$, and it is obvious that

(139) $dxdy = \text{FGSK}$.

Now if A = area of the curve, $\therefore\ d\text{A} = \text{DESN}$, as was shown in Proposition I.

But FGSK may be regarded as the differential of DESN, and is consequently the differential of a differential, or the second differential of the area, that is,

(140) $$d^2\text{A} = dxdy.$$

If this be integrated twice, we have,

(141) $$A = \iint dxdy.$$

By inspecting the figure, it is evident that FG or dx may remain constant while dy or KF is repeated along ND. Hence one of the integrations of (140) may be made for dx constant, and dy variable. This gives,

(142) $$dA = ydx.$$

If now the equation of the curve be,

(143) $$y = \phi x,$$

the constant added to (142) may be eliminated by taking (142) between the limits $y = o$, $y = \varphi x$, by which (142) becomes,

(144) $$dA = \varphi xdx.$$

Equation (144) integrated, gives,

(145) $$A = \int \varphi xdx = \varphi' x + C,$$

denoting the integral of φxdx by $\varphi' x$. If the constant in (145) be eliminated by taking that integral between the limits $x = o$, $x = AC$, we have the whole area, ACB.

The integration of (140) for dx constant and dy variable, may be regarded as the addition of the elemental rectangles KG, in the differential area DS.

Again, by inspecting the figure, it is evident that ML or dy may remain constant while FG or dx is repeated along LR. Hence one of the integrations of (140) may be made for dy constant and dx variable. This gives,

(146) $$dA = xdy.$$

If now the equation of the curve be,

(147) $$x = \psi y,$$

the constant added in (146) may be eliminated by taking it between the limits $x = o$, $x = \psi y$, by which (146) becomes,

(148) $$dA = \psi ydy.$$

This integrated again, gives,

(149) $$A = \int \psi ydy = \psi' y + C,$$

where $\psi' y$ is put for the integral of ψydy. If the constant be determined by taking this integral between the limits $y = o$, and $y = AB$, we get the area between the curve and the axis of y. The integration of (140) for dx variable and dy constant, may be regarded as

the addition of the elemental rectangles KG in the differential area MR. The integral (145) is obviously the same in signification as (113), and the integral (149) the same as (115).

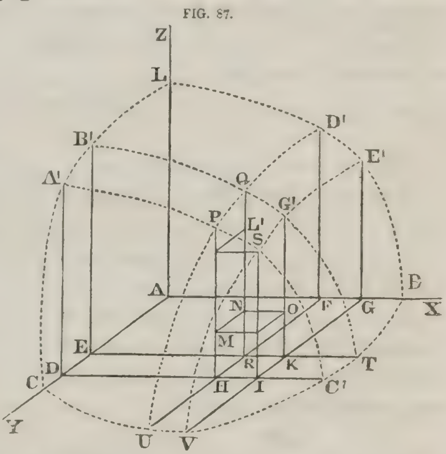
The object of this Proposition has been to familiarise the student with the Geometry of Double Integrals, as preparatory to the succeeding investigations.

PROPOSITION IX.

Determine the volume of any regular solid.

Proposition V. enables us to find the volume of a *Solid of Revolution*, but the object of the present Proposition is to deduce a method applicable to any given solid.

FIG. 87.



Let the equation of the surface LCB be

(150) $z = \varphi\,(x,y)$.

Let CB, the intersection of the surface with the plane xy, be represented by the equation,

(151) $y = \psi x$.

Let (x,y,z) be the point Q on the surface. Then taking the point S indefinitely near to Q, and passing through each of the points Q and S two parallel planes, we have the figure, in which,

$$\text{(A)}\quad \begin{cases} AF = x, \\ FR = y, \\ RQ = z, \end{cases} \qquad \text{and} \qquad \begin{aligned} RK &= dx, \\ RH &= dy, \\ RN &= QL' = dz. \end{aligned}$$

If V be the volume comprised between the co-ordinate planes, and the parallel planes D′FU and B′ET, then,

(152) $$dV = \text{solid } D'QRKGE',$$
and the part QPHKG′ is the differential of the solid D′QRKGE′, or,
(153) $$d^2V = \text{solid } QPHKG',$$
and the solid HKOM is the differential of the solid QPHKG′, or,
(154) $$d^3V = \text{solid } HKOM.$$

But it is obvious that by notation (A),
(155) $$\text{solid } HKOM = dxdydz. \qquad \text{Hence,}$$
(156) $$d^3V = dxdydz.$$

The third integral of this is,
(157) $$V = \iiint dxdydz.$$

By inspecting the figure, it is obvious that dx and dy may remain constant, while dz is repeated along the line QR. That is, (156) may be integrated for z variable, and dx and dy constant. Integrating (156) once on this hypothesis, it becomes,
(158) $$d^2V = dxdyz,$$
or, which is the same,
(159) $$\frac{d^2V}{dxdy} = z.$$

This may be corrected by taking it between the limits $z = o$, $z = \varphi\,(x,y)$, [see (150),] which gives,
(160) $$\frac{d^2V}{dxdy} = \varphi\,(x,y),$$
which is the whole line QR.

By the figure, it is obvious that dx may be constant, while dy is repeated along the line FU. Hence (160) may be integrated for dx constant and dy variable. Multiply (160) by dy, and integrating for y variable, we have,
(161) $$\frac{dV}{dx} = \int \varphi(x,y)\,dy = \varphi'(x,y) + C,$$
where $\varphi'(x,y)$ is put for the integral of $\varphi(x,y)\,dy$.

This may be corrected by being taken between the limits $y = o$, $y = \psi x$, [see (151),] which changes (161) to
(162) $$\frac{dV}{dx} = \varphi'(x,\psi x).$$

Multiply this by dx, and integrate, and we have,

(163) $$\therefore \quad V = \int \varphi'(x,\psi x)\, dx = \varphi'' x + C,$$

which may be corrected by taking it between the limits $x = o$, $x = AB$, or any other assigned limits.

It is obvious that by taking (158) between the limits $z = o$, $z = \varphi(x,y)$, we get the solid whose lower base is the parallelogram IIK, and upper base the surface QS. It is also obvious that by integrating (160) for y, as the only variable, we get the area of the parallel section D'FU, which area is the second side of (162). And after multiplying (162) by dx, in order to integrate, we see that the solid is the parallel area D'FU multiplied by dx, or FG. Hence we may regard the area D'FU as the generator of the solid, that area moving parallel to itself, and varying so as always to be a section of the surface.

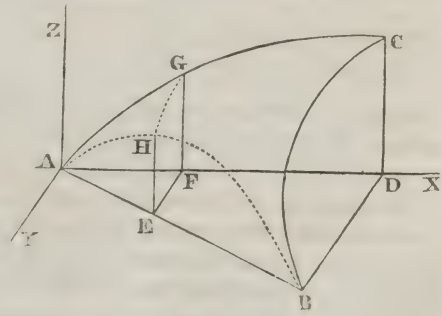
Instead of integrating (156) for z, as the only variable, we might have integrated it for x or y, as the only variable. In (160) also, we might have integrated first for x instead of y.

In practice, it is usual to begin with (159), or rather with (160), its equal.

Illustrative of this theory, take the following:

Example 1. A paraboloid of revolution is cut into a wedge by two planes, whose line of intersection passes through the vertex, and is perpendicular to the axis of revolution, the base of the paraboloid is the back of the wedge, find its solid content.

FIG. 88.



Let AX be the axis of revolution.

Let the planes that cut the paraboloid be perpendicular to XY. AZ is their line of intersection, and CABD is one-fourth of the wedge. Let the equation of AB, the trace of one of the cutting planes on the plane XY, be

$$(a) \qquad y = ax,$$

and the equation of the paraboloid,

$$(b) \qquad z = (px - y^2).$$

This value of z, put into (159), we have,

$$(c) \qquad \frac{d^2V}{dxdy} = (px - y^2)^{\frac{1}{2}}.$$

Multiply by dy, and integrating for x constant, we have,

$$(d) \qquad \frac{dV}{dx} = \int (px - y^2)^{\frac{1}{2}} dy.$$

The second side of (d), for x constant, is integrated at (80), from which we have,

$$(e) \qquad \frac{dV}{dx} = \frac{y}{2}(px - y^2)^{\frac{1}{2}} + \frac{px}{2}\sin.^{-1}\frac{y}{\sqrt{px}}.$$

This taken between the limits $y = o$, $y = ax$, we have,

$$(f) \qquad \frac{dV}{dx} = \frac{ax}{2}(px - a^2x^2)^{\frac{1}{2}} + \frac{px}{2}\sin.^{-1}a\sqrt{\frac{x}{p}}.$$

Multiply by dx, and integrating, we have,

$$(g) \qquad V = \frac{a}{2}\int (p - a^2x)^{\frac{1}{2}}x^{\frac{3}{2}}dx + \frac{p}{2}\int x \sin.^{-1}a\sqrt{\frac{x}{p}}.\,dx,$$

the first term of which fulfils the second condition of integrability, (66). The second term may be integrated by parts.

For the whole quarter of the wedge, (g) must be taken between the limits $x = o$, $x = \frac{p}{a^2}$.

Ex. 2. A given cylinder is cut by two planes into a wedge, having a diameter of one end of the cylinder for its edge, and the area of the other end for its back, find the volume of the wedge.

Ans. $V = MR^2 (\pi - \frac{4}{3})$,

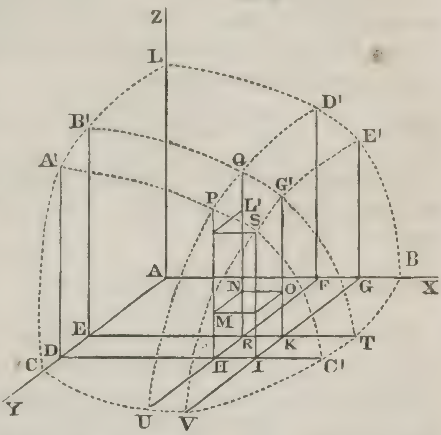
where M = axis of the cylinder, and R = radius of the base.

Since πMR^2 is the volume of the cylinder, $\frac{4}{3}MR^2$ is the volume of the parts cut off in forming the wedge. The volume of these parts is therefore a geometrical cube.

PROPOSITION X.

Determine the area of any regular surface.

FIG. 89.



Let CLB be the surface, and its equation

(164) $\quad z = \varphi(x,y)$.

Let (x,y,z) be the point Q, and

(165) $\quad y = \psi x$,

be the intersection CB of the surface with the plane XY. If S represent the surface B′LD′Q, and S and Q being indefinitely near each other, D′QG′E′ is the differential of the surface, that is, dS = D′E′G′Q, and the differential of D′E′G′Q is QG′SP, that is, d^2S = QG′SP. But as QG′SP is indefinitely small, it may be regarded as a plane area, and it is obvious that

$dxdy =$ HK, is its projection on plane XY, and similarly,

$dxdz =$ its projection on plane ZX, and

$dydz =$ its projection on plane ZY.

But, as is shown in books on Analytical Geometry, the square of an area in space equals the sum of the squares of its projections on three co-ordinate planes. Hence,

(166) $\quad (\mathrm{PQG'S})^2 = (d^2\mathrm{S})^2 = dx^2dy^2 + dx^2dz^2 + dy^2dz^2$.

Divide this by dx^2dy^2, and extracting the root, we have,

$$(167) \qquad \frac{d^2\mathrm{S}}{dxdy} = \left(1 + \frac{dz^2}{dy^2} + \frac{dz^2}{dx^2}\right)^{\frac{1}{2}}.$$

This is the second differential of the surface.

The partial differential coefficients under the root can be obtained from (164) in terms of x and y. Hence the second side of (167)

is a function of x and y. Represent it by $\pi(x,y)$, and we may write (167) in the form,

$$(169) \qquad \frac{d^2\mathrm{S}}{dxdy} = \pi(x,y).$$

It is obvious, that in the surface, x may vary, while y remains constant, and conversely. Hence we may integrate (169) for one of the co-ordinates x,y constant, and the other variable. Integrate it for y variable, and we have,

$$(170) \qquad \frac{d\mathrm{S}}{dx} = \int \pi(x,y)\, dy = \pi'(x,y)$$

This may be corrected by being taken between any given limits, as $y = o$, and $y = \psi x$. [See (165).] By these limits, (170) becomes,

$$(170a) \qquad \frac{d\mathrm{S}}{dx} = \pi'(x,\psi x).$$

This may be integrated for the variable x, and we have,

$$(171) \qquad \mathrm{S} = \int \pi'(x,\psi x)\, dx = \pi'' x.$$

This may be corrected by being taken between any given limits, as $x = o$, $x = \mathrm{AB}$. In (171), which gives the surface CLB, we obviously have the surface as a function of x. By integrating (169) first for x variable, we would get the surface as a function of y.

Ex. 1. Find the surface of the wedge of Example 1, last Proposition.

Referring to fig. 88, we have for the equation of the surface,

$$(a) \qquad z^2 + y^2 = px,$$

and for the equation of the plane AHB,

$$(b) \qquad y = ax.$$

From (a) we get

$$(c) \qquad \frac{dz^2}{dx^2} = \frac{p^2}{4px - 4y^2}, \quad \text{and} \quad \frac{dz^2}{dy^2} = \frac{y^2}{px - y^2}.$$

These values put into (167), we get for the particular form (169),

$$(d) \qquad \frac{d^2\mathrm{S}}{dxdy} = \frac{(4px + p^2)^{\frac{1}{2}}}{2\,(px - y^2)^{\frac{1}{2}}}.$$

Integrate this for y variable, we have,

$$(e)\qquad \frac{dS}{dx}=\frac{(4px+p^2)^{\frac{1}{2}}}{2}\int\frac{dy}{(px-y^2)^{\frac{1}{2}}}=\frac{(4px+p^2)^{\frac{1}{2}}}{2}\sin.^{-1}\frac{y}{\sqrt{px}}.$$

This corrected by the limits $y = o$, $y = ax$, we have,

$$(f)\qquad \frac{dS}{dx}=\frac{(4px+p^2)^{\frac{1}{2}}}{2}\sin.^{-1} a\sqrt{\frac{x}{p}}.$$

Multiply by dx, and integrate, and we have,

$$(g)\qquad S=\int\frac{(4px+p^2)^{\frac{1}{2}}}{2}dx\sin.^{-1} a\sqrt{\frac{x}{p}},$$

a function of x, which may be integrated by parts. Equation (g) corrected by the limits $x = o$, $x = \frac{p}{a^2}$, gives the whole convex surface of one-fourth of the wedge.

Ex. 2. Determine the surface of the cylindric wedge of Example 2, last Proposition.

Ans. $S = 2\pi RM - 4RM$,

where M = axis of the cylinder, and R = radius of its base. Since $2\pi RM$ is the surface of the cylinder, $4RM$ must be the convex surface of the parts cut off in forming the wedge, which is therefore quadrable.

The last two Propositions are particularly applicable in finding the volume and surface of twisted surfaces.

PROPOSITION XI.

Determine the length of a curve of double curvature.

Let (x,y,z) be any point P on a curve of double curvature. Let S be the length of the curve measured from any given point. Then if the curve be regarded as a polygon of an indefinite number of sides, we may represent one of these sides by dS, and taking dS as the diagonal of a parallelopipedon, whose edges are dx, dy and dz, we have, by the relation between the diagonal and edges of a parallelopipedon,

$$(172)\qquad dS = (dx^2 + dy^2 + dz^2)^{\frac{1}{2}}.$$

Integrating this, we have,

(173) $$S = \int (dx^2 + dy^2 + dz^2)^{\frac{1}{2}}.$$

If now the equation of the curve be represented by

(174) $$x = \phi z, \qquad y = \pi z,$$

it is obvious that the values of dx and dy deduced from (174), will be functions of z. Hence by means of (174), (173) may be expressed in terms of a single variable, and integrated.

Ex. 1. Determine the length of the curve,

(m) $$x = az^2, \qquad y = bz^2.$$

Differentiating equation (m), we have,

(n) $$\therefore \quad dx^2 = 4a^2z^2dz^2, \quad \text{and} \quad dy^2 = 4b^2z^2dz^2.$$

These values put into (173), it becomes,

(p) $$S = \int (1 + c^2z^2)^{\frac{1}{2}}dz,$$

where $c^2 = 4a^2 + 4b^2$.

Equation (p) is integrated at (79), where $a = 1$, and $b^2 = c^2$.

Ex. 2. Determine the length of the curve,

$$z^2 = \text{R}x, \qquad x^2 + y^2 = \text{R}x.$$

PROPOSITION XII.

Determine the integral by means of elliptic or hyperbolic arcs.

Suppose we have to integrate the expression,

(175) $$\frac{\text{A}x^2dx}{(\text{P}x^2 - x^2 - \text{Q})^{\frac{1}{2}}}.$$

If we take as the equation of the ellipse,

(176) $$a^2y^2 + b^2x^2 = a^2b^2,$$

the differential of the arc of the ellipse is, by (119),

(177) $$dz = \frac{[a^4 - (a^2 - b^2)\,x^2]^{\frac{1}{2}}dx}{a\,(a^2 - x^2)^{\frac{1}{2}}}.$$

If we assume

(178) $$[a^4 - (a^2 - b^2)\,x^2]^{\frac{1}{2}} = ax',$$

and solve this for x, we have,

(179) $$x = \frac{a\,(a^2 - x'^2)^{\frac{1}{2}}}{(a^2 - b^2)^{\frac{1}{2}}}.$$

Put the values of x and dx from (179) into (177), and we have,

$$(180) \qquad dz = \frac{x'^2 dx'}{[(a^2+b^2)\,x'^2 - x'^4 - a^2b^2]^{\frac{1}{2}}}.$$

This is the same in form as (175). Hence (175) is the arc of an ellipse. To find the axes of the ellipse, equate the like powers of x under the radicals of (180) and (175), and we have,

$$a^2 + b^2 = \mathrm{P}, \qquad \text{and} \qquad a^2b^2 = \mathrm{Q},$$

from which the axes a and b may be found. The abscissa of the extremity of the elliptic arc (175), is the second side of (179).

Again, suppose we have to integrate

$$(181) \qquad \frac{\mathrm{A}dx}{x^2\,(\mathrm{P}x^2 - x^4 - \mathrm{Q})^{\frac{1}{2}}}.$$

Assume the numerator of (177)

$$(182) \qquad [a^4 - (a^2 - b^2)\,x^2]^{\frac{1}{2}} = \frac{a^2b}{x'}.$$

From which we get,

$$(183) \qquad x = \frac{a^2\,(x'^2 - b^2)^{\frac{1}{2}}}{x'\,(a^2 - b^2)^{\frac{1}{2}}}.$$

This value of x, put into (181), changes (181) to

$$(184) \qquad dz = \frac{-\,a^2b^2dx'}{x'^2\,[(a^2+b^2)\,x'^2 - x'^4 - a^2b^2]^{\frac{1}{2}}}.$$

Whence we see that (181) is an elliptic arc, the abscissa of whose extremity is the value of x in (183), and whose axes are found as before.

Again, suppose we have to integrate,

$$(185) \qquad \frac{\mathrm{A}x^2dx}{(\mathrm{P}x^2 + x^4 - \mathrm{Q})^{\frac{1}{2}}}.$$

If we take the equation of the hyperbola,

$$(186) \qquad a^2y^2 - b^2x^2 = -\,a^2b^2,$$

the differential of the arc of the hyperbola is, by (119),

$$(187) \qquad dz = \frac{[(a^2+b^2)\,x^2 - a^4]^{\frac{1}{2}}dx}{a\,(x^2 - a^2)^{\frac{1}{2}}}.$$

Assume

$$(188) \qquad (a^2+b^2)\,x^2 - a^4 = a^2x'^2,$$

and (187) becomes, in terms of x',

$$(188a) \qquad dz = \frac{x'^2 dx'}{[(a^2 - b^2)\, x'^2 + x'^4 - a^2b^2]^{\frac{1}{2}}},$$

which is of the same form as (185). Hence (185) is the arc of a hyperbola, whose axes are found as in the elliptic arc (180).

Again, suppose we have to integrate the form,

$$(189) \qquad \frac{A dx}{x^2\,(Px^2 - x^4 + Q)^{\frac{1}{2}}}.$$

Assume

$$(190) \qquad (a^2 + b^2)\, x^2 - a^4 = \frac{a^2b^2}{x'^2},$$

and (187) becomes of the same form as (189), which is therefore the arc of a hyperbola determined as before.

PROPOSITION XIII.

Determine the integral of a differential equation of the first order and degree, containing two variables.

We have hitherto integrated particular expressions, or those explicit equations in which only one side required attention. We now propose to integrate implicit differential equations of two variables.

An implicit differential equation is one involving the variables and their differentials in *any* manner.

The arrangement of the terms and factors of a differential equation in such a manner that one side of the equation contains only one of the variables, and the other the other, is called the *Separation of the Variables.*

To integrate a differential equation, separate the variables, and integrate each side of the equation. The following are some of the cases in which the variables can be separated.

Case 1.

Let the equation be of the form,

$$(191) \qquad Fy.dx + fx.dy = o.$$

This is at once separated by transposing and dividing by $Fy.fx$, by which we have,

(192) $$\frac{dx}{fx} = -\frac{dy}{Fy},$$

where each side is to be integrated separately.

As an example, take the following equation,

(*a*) $$aydx - x^2dy = o.$$

This at once becomes,

(*b*) $$a\frac{dx}{x^2} = \frac{dy}{y} \cdots.$$

Integrating both sides of (*b*), we have,

$$-\frac{a}{x} = \log.y + C.$$

Case 2.

Let the equation be of the form,

(193) $$fx.Fy.dx + f'x.F'y.dy = o.$$

This is separated by dividing by $Fy.f'x$, by which we have,

(194) $$\frac{fx\,dx}{f'x} = -\frac{F'y\,dy}{Fy}.$$

As an example, take the following equation.

$$cx^2ydx - (x-a)\,y^2dy = o.$$

Divide by $y - (x - a)$, and the variables are separated, after which each side is readily integrated.

Case 3.

When the equation is homogeneous in respect of the variables, they can be separated. Take the form,

(195) $$x^ny^mdx + Ax^a\,y^cdy = o,$$

in which $n + m = a + c = p$.

To separate the variables of (195), assume the equation,

(196) $$y = zx,$$

and putting this value into (195), it becomes,

(197) $$x^pz^mdx + Ax^pz^cdy = o.$$

Divide this by x^p, substitute for dy its value from differentiating (196), and we have,

(198) $$z^m dx + Az^c(zdx + xdz) = o,$$
or, by multiplying and transposing,

(199) $$(z^m + Az^{c+1})\,dx = -Az^c xdz.$$

From this we have,

(200) $$\frac{dx}{x} = -\frac{Az^c dz}{z^m + Az^{c+1}},$$

each side of which may be integrated separately.

Instead of (196) we might obviously put $x = zy$, and obtain a similar result.

As an example, take the following equation:

(d) $$2xydx + y^2dy - x^2dy = o.$$

This is homogeneous, the degree of homogeneity being 2. If we put

(e) $$y = zx,$$

then $dy = zdx + xdz$, and (d) becomes,

(f) $$\frac{dx}{x} = \frac{dz\,(1 - z^2)}{z + z^3}.$$

If we put $x = zy$, (d) becomes,

(g) $$\frac{dy}{y} = -\frac{2zdz}{1 + z^2},$$

which is a simpler form. The integral of (g) is,

(h) $$\log.y = -\log.(1 + z^2) + \log.c = \log.\left(\frac{c}{1 + z^2}\right).$$

Removing the logarithms, and restoring the value of z, (h) becomes,

(k) $$x^2 + y^2 - cy = o, \text{ a circle.}$$

Ex. 2. Integrate $xdy - ydx = (x^2 + y^2)^{\frac{1}{2}}dx$.

Ans. $x^2 = 2cy + c^2$.

Ex. 3. Integrate $y^2dy = 3xydx - x^2dy$.

Ans. $y^2 = 2x^2 + Cy^{\frac{2}{3}}$.

Case 4.

An equation of the form,

(201) $$(a + bx + cy)\,dx = (m + nx + py)\,dy,$$

may be rendered homogeneous, and integrated by assuming

$$x = x' + h, \quad \text{and} \quad y = y' + k.$$

Putting these values into (201), it becomes,

(202) $(a+bh+ck+bx'+cy')dx' = (m+nh+pk+nx'+py')\,dy'$.

As we introduced two indeterminate quantities, h and k, we may assume in (202) any two arbitrary conditions to determine these. Assume the conditions

(203) $a + bh + ck = o$, and $m + nh + pk = o$.

Solve equations (203) for h and k, and substitute these values into (202), and it becomes

(204) $(bx' + cy')\,dx' = (nx' + py')\,dy'$,

a homogeneous equation, which may be integrated by Case 3.

Case 5.

Let the equation be of the form,

(205) $dy + \mathrm{P}ydx = \mathrm{Q}dx$,

in which P and Q are functions of x. Assume $y = z\mathrm{R}$, R being a function of x. Differentiating this, we have, $dy = zd\mathrm{R} + \mathrm{R}dz$. These values of y and dy put into (205), it becomes,

(206) $zd\mathrm{R} + \mathrm{R}dz + \mathrm{PR}zdx = \mathrm{Q}dx$.

The quantity R being an indeterminate coefficient, we may determine it by assuming any arbitrary condition involving R. To reduce (206), assume

(207) $\mathrm{R}dz + \mathrm{PR}zdx = o$.

This reduces (206) to

(208) $zd\mathrm{R} = \mathrm{Q}dx$.

Equation (207) may be divided by R, and the variables being separated, we have,

(209) $\dfrac{dz}{z} = -\,\mathrm{P}dx, \quad \therefore \quad \log. z = -\int \mathrm{P}dx$,

or, passing to numbers, [see equations (15), (16),]

(210) $z = e^{-\int \mathrm{P}dx}$.

This value of z put into (208), we have,

(211) $d\mathrm{R} = e^{\int \mathrm{P}dx}\mathrm{Q}dx, \quad \therefore \quad \mathrm{R} = \int e^{\int \mathrm{P}dx}\mathrm{Q}dx + \mathrm{C}$.

The values of R and z from (211) and (210), put into the equation $y = z\mathrm{R}$, we have,

(212) $y = e^{-\int \mathrm{P}dx}\left(\int e^{\int \mathrm{P}dx}\mathrm{Q}dx + \mathrm{C}\right)$,

which is the integral of (205).

The form (205) is in some books called *a Linear Equation*, because it contains y only in the first power.

Ex. 1. Integrate the equation,

$$(a) \qquad dy + ydx = ax^3dx.$$

Here $P = 1$, $Q = ax^3$, and (212) becomes,

$$(b) \qquad y = a\,(x^3 - 3x^2 + 6x - 6) + Ce^{-x},$$

the integral of (a). The form,

$$(213) \qquad y^{m-1}dy + Py^m dx = Qy^n dx,$$

may be reduced to the form (205); for dividing (213), by y^n, and putting $y^{m-n} = (m - n)\,z$, (213) becomes,

$$(214) \qquad dz + (m - n)\,Pzdz = Qdx,$$

or putting P' for $(m - n)\,P$, (214) becomes,

$$(215) \qquad dz + P'zdx = Qdx.$$

This is the same form as (205).

Ex. Integrate the equation,

$$(c) \qquad dy + ydx = xy^3dx.$$

This is of the form (213), and the integral is,

$$\frac{1}{y^2} = Ce^{2x} + x + \frac{1}{2}.$$

Suppose we have to integrate the form,

$$(216) \qquad x^n dy + by^p dx = gx^m dx.$$

This equation may sometimes be rendered homogeneous by assuming

$$(217) \qquad y = z^c, \quad \therefore \quad dy = cz^{c-1}dz, \quad \text{and} \quad y^p = z^{cp}.$$

These values put into (216), it becomes,

$$(218) \qquad cx^n z^{c-1}dz + bz^{cp}dx = gx^m dx.$$

This is homogeneous if

$$(219) \qquad n + c - 1 = cp = m.$$

If $n = o$, $m = o$, and $p = 2$, (216) becomes,

$$(220) \qquad dy + by^2dx = gdx, \quad \therefore \quad dx = \frac{dy}{g - by^2},$$

where the variables are separated.

If $n = o$, and $p = 2$, (216) becomes,

$$(221) \qquad dy + by^2dx = gx^m dx.$$

This is called the equation of *Riccati*.

For (221), the conditions (219) become

(222) $\quad c - 1 = 2c = m, \quad \therefore \quad c = -1$, and $m = -2$.

Hence (221) may be rendered homogeneous, if $m = -2$.

Besides the particular case $m = -2$, the variables in (221) may be separated for some other values of m, but we will not investigate the subject at present. Equation (221), and the more general form, (216), has occupied much attention among writers on the Integral Calculus, but no general method of separating the variables in (221) has yet been devised.

Before proceeding further with differential equations, we will give a number of Geometrical Applications in which such equations are employed. And in order that the student may become familiar with these equations, and with the application of the Integral Calculus to Geometry, we will commence with very simple problems, and afterwards proceed to those that are more difficult.

APPLICATION 1st.

Determine the curve whose subnormal is constant.

Put the value of the subnormal, [given in Differential Calculus, Proposition VI., (73),] equal to a constant quantity m, and we have,

(223) $$\frac{ydy}{dx} = m.$$

This is the differential equation of the problem.

Separating the variables, we have,

(224) $$ydy = mdx,$$

and integrating, we have,

(224a) $$y^2 = 2mx + C.$$

Hence the curve possessing the proposed property is the parabola, with the origin on the axis of abscissas. The constant C, in (224a), need not be determined for the solution of the Proposition. It merely defines the position of the origin, and the curve, whatever C may be, is a parabola. To determine C would require another condition.

APPLICATION 2d.

Determine the curve whose subnormal varies as the square of the abscissa.

Using the value of the subnormal given in Differential Calculus, equation (73),

$$\frac{ydy}{dx} = mx^2, \tag{225}$$

is the problem put into equation.

Separating the variables, and integrating, we have,

$$\frac{y^2}{2} = \frac{mx^3}{3} + C, \tag{226}$$

the curve required, which is the semicubical parabola. The constant C is not determinable by the given condition of the problem. It merely fixes the position of the origin.

APPLICATION 3d.

Determine the curve whose subnormal varies as a given function of the abscissa.

Denoting the given function by φx, we have, by the condition of the problem,

$$\frac{ydy}{dx} = \varphi x, \tag{227}$$

the differential equation of the required curve. Integrating, we have,

$$y^2 = 2\int \varphi x dx, \tag{228}$$

where, for φx, we may take any combination of x.

If $\varphi x = mx^2$, (227) becomes (225), and the curve is the semicubical parabola (226).

APPLICATION 4th.

Determine the curve whose subnormal varies as a given function of the ordinate.

Denoting the given function of the ordinate by φy, we have,

$$\frac{ydy}{dx} = \varphi y, \tag{229}$$

for the differential equation of the curve.

$$\therefore \int \frac{ydy}{\varphi y} = x + C, \tag{230}$$

where for φy we may take any combination of y.

If $\varphi y = my^2$, (230) becomes $\frac{1}{m}$ $\log.y = x + \mathrm{C}$, which is the logarithmic curve.

APPLICATION 5th.

Determine the curve whose subnormal varies as a given function of the co-ordinates of the normal point.

Denoting the given function of the co-ordinates by $\varphi(x,y)$, we have,

$$\frac{ydy}{dx} = \varphi(x,y), \tag{231}$$

for the differential equation. To integrate this, take as a particular case, $\varphi(x,y) = mx^2y^3$, and (231) becomes,

$$\frac{dy}{y^2} = mx^2dx, \quad \therefore -\frac{1}{y} = \frac{mx^3}{3} + \mathrm{C}, \text{ is the curve.}$$

If $\varphi(x,y) = m\,(x + ny)$, (231) is,

$$y\,\frac{dy}{dx} = m\,(x + ny), \tag{232}$$

a homogeneous equation, which may be treated by Case 3, last Proposition.

If in the last five applications, we take the subtangent instead of the subnormal, we find curves whose subtangent possesses given properties.

APPLICATION 6th.

Determine the curve whose subtangent is constant.

Take for the subtangent the expression given at equation (69), of the Differential Calculus, and we have,

$$y\frac{dx}{dy} = m, \tag{233}$$

for the differential equation of the curve.

Integrate (233), and we have for the curve, $m \log.y = x + \mathrm{C}$. Hence the logarithmic curve has a constant subtangent. In like manner, determine other curves whose subtangent has given properties.

APPLICATION 7th.

Determine the curve whose normal is constant.

Employing the value of the normal line given at equation (75) of the Differential Calculus, we have,

$$(234)\qquad \left(y^2 + y^2 \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = m,$$

for the differential equation of the curve.

From this we readily get,

$$(235)\qquad dx = (m^2 - y^2)^{-\frac{1}{2}} y dy, \quad \therefore \quad x = (m^2 - y^2)^{\frac{1}{2}} + C,$$

or, $(x - c)^2 + y^2 = m^2$, which shows the curve to be a circle with the origin on the axis of abscissas.

APPLICATION 8th.

Determine the curve whose normal varies as a given function of the ordinate of the normal point.

Denoting the given function of the ordinate by φy, we have,

$$(236)\qquad \left(y^2 + y^2 \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = \varphi y,$$

for the differential equation of the curve. From this we get, by separating the variables,

$$(237)\qquad dx = y dy \left[(\varphi y)^2 - y\right]^{2-\frac{1}{2}}.$$

By putting for φy any combination of y, this may be integrated, and the curve determined.

APPLICATION 9th.

Determine the curve whose tangent line is constant.

Using the expression for the length of the tangent line, at (74), Differential Calculus, we readily form the differential equation.

APPLICATION 10th.

Determine the curve whose tangent line varies as a given function of the ordinate of the point of tangency.

Denote by φy, the given function of the ordinate, and employing the length of the tangent at (74), Differential Calculus, we readily form the differential equation.

APPLICATION 11th.

Determine the curve whose normal equals the abscissa increased by the subnormal.

(238) $$y^2 + y^2 \frac{dy^2}{dx^2} = \left(y \frac{dy}{dx} + x\right)^2,$$

is the differential equation. From this we have,

(239) $$y^2dx - 2xydy - x^2dx = o.$$

This is a homogeneous equation, which may be integrated by Case 3, Proposition XIII, and from which we have,

(240) $$y^2 + x^2 - 2cx = o.$$

This is a circle with the origin on the perimeter, and centre on the axis of x.

APPLICATION 12th.

Determine the curve in which the subnormal is to the subtangent as the ordinate is to the abscissa of the point of tangency.

By the problem, we have,

(241) $$\frac{ydy}{dx} : y\frac{dx}{dy} :: y : x.$$

from which we get,

(242) $$y^{-\frac{1}{2}}dy = x^{-\frac{1}{2}}dx,$$

the differential equation. Integrate this, and we have,

(243) $$y^{\frac{1}{2}} = x^{\frac{1}{2}} + \frac{C}{2},$$

for the curve. Clearing this of radicals, we have a parabola with the co-ordinate axes tangent to the curve, and the origin at the intersection of the axis and directrix.

APPLICATION 13th.

Determine the curve in which the subnormal is to the subtangent as the abscissa to the ordinate of the point of tangency.

By the problem, we have,

(244) $$\frac{ydy}{dx} : \frac{ydx}{dy} :: x : y,$$

from which we have,

(245) $$y^{\frac{1}{2}}dy = x^{\frac{1}{2}}dx, \quad \therefore \quad y^{\frac{3}{2}} = x^{\frac{3}{2}} + \text{C},$$

which is a line of the 6th order.

APPLICATION 14th.

Determine the curve in which the area varies as the ordinate.

Take the value of the area at (113), Proposition I., and we have,

(246) $$\int ydx = my,$$

for the equation of the curve.

In order to integrate an equation involving an integral, as (246), we must dispose of the sign of integration by first differentiating the equation. Differentiate (246), and we have,

(247) $$ydx = mdy.$$

This is the differential equation of the curve required.

Integrate this, and we have,

(248) $$x = m \log.y + \text{C}$$

for the curve required.

APPLICATION 15th.

Determine the curve in which the area varies as a given function of the ordinate.

Denote the given function of the ordinate by ϕy, and we have,

(249) $$\int ydx = \phi y,$$

for the equation. Differentiate this, and we have,

(250) $$ydx = \phi' ydy,$$

where $\phi' y$ is put for what ϕy becomes when differentiated.

Separate the variables in (250), and it may be integrated for any given value of ϕy.

APPLICATION 16th.

Determine the curve whose area varies as the product of the co-ordinates.

By the problem, we have,

(251) $$\int ydx = mxy,$$

for the equation of the curve. Differentiate this, and we have,

(252) $$(y - my)\, dx = mxdy,$$

which belongs to Case 1, Proposition XIII.

Separate the variables, and we have,

(253) $$\frac{dx}{x} = \frac{m}{1-m} \cdot \frac{dy}{y} \quad \therefore \quad \log.x = \frac{m}{1-m} \log.cy,$$

or removing the logarithms, [see equations (15), (16),] we have,

(254) $$x^{1-m} = (cy)^m.$$

If $m = \frac{2}{3}$, this becomes,

(255) $$x = (cy)^2,$$

the common parabola.

APPLICATION 17th.

Find the curve in which the area varies as the sum of the co-ordinates.

Express this law of variation by $m\,(x + y)$, and we have, by the problem,

(256) $$\int ydx = m\,(x + y),$$

for the equation. Integrate this, and we have,

(257) $$x = m \log.(y - m) + C,$$

for the curve.

APPLICATION 18th.

Find the curve in which the arc varies as the $\frac{3}{2}$ power of the abscissa.

Take the general form, for the length of an arc, at Proposition III., (119), and we have, by the problem,

(258) $$\int (dx^2 + dy^2)^{\frac{1}{2}} = mx^{\frac{3}{2}}.$$

Differentiate this to dispose of the sign of integration, and we have,

$$(dx^2 + dy^2)^{\frac{1}{2}} = \tfrac{3}{2}\, mx^{\frac{1}{2}} dx.$$

Squaring, transposing and extracting the root, we have,

(259) $$dy = \left(\frac{9}{4}\, m^2x - 1\right)^{\frac{1}{2}} dx.$$

Integrate this, and we get,

$$(260) \qquad y = \frac{8}{27m^2}\left(\frac{9}{4}m^2x - 1\right)^{\frac{3}{2}} + C,$$

the semicubical parabola.

APPLICATION 19th.

Determine the curve in which the arc varies as a given function of the ordinate, or abscissa, or of the co-ordinates.

By the Proposition, we have,

$$(261) \qquad \int (dx^2 + dy^2)^{\frac{1}{2}} = \phi y, \qquad \text{or,}$$

$$(262) \qquad \int (dx^2 + dy^2)^{\frac{1}{2}} = \varphi x, \qquad \text{or,}$$

$$(263) \qquad \int (dx^2 + dy^2)^{\frac{1}{2}} = \phi(x,y),$$

according as the arc varies as a function of the ordinate, or abscissa, or co-ordinates.

For any given value of ϕy, or ϕx, or $\phi(x,y)$, these equations can be differentiated, to dispose of the sign of integration, and if the variables are then separable, the integration may be effected, and the curve determined.

APPLICATION 20th.

Determine the curve whose surface of revolution varies as a given function of the ordinate.

In Proposition IV., (127), we have the expression for the surface of revolution. Putting that expression equal to a given function of the ordinate, which we will denote by φy, we have,

$$(264) \qquad 2\pi\int y\,(dx^2 + dy^2)^{\frac{1}{2}} = \varphi y.$$

Differentiate this, and we have,

$$(265) \qquad 2\pi y\,(dx^2 + dy^2)^{\frac{1}{2}} = \varphi' y dy.$$

From which we get,

$$(266) \qquad dx = \frac{[\,(\varphi' y)^2 - 4\pi^2 y^2]^{\frac{1}{2}}\,dy}{2\pi y}.$$

This may be integrated for any particular value of φy.

APPLICATION 21st.

Determine the curve whose volume of revolution varies as a given function of either or both of the co-ordinates.

By Proposition V., (130), we have the expression for the volume of such solids. Then by the problem, we have,

(267) $\pi \int y^2 dx = \phi x, \quad \pi \int y^2 dx = \varphi y, \quad \pi \int y^2 dx = \phi(x,y).$

These equations may be integrated for any particular forms of the given functions.

APPLICATION 22d.

Determine the curve whose polar subtangent varies as a given function of the radius vector.

In Differential Calculus, Proposition XXXIII., (172), we have the general form of the polar subtangent, which put equal to φr, a given function of the radius vector, and we have,

(268) $$\frac{r^2 d\omega}{dr} = \varphi r, \quad \therefore \quad d\omega = \frac{\varphi r dr}{r^2},$$

which for any particular value of φr, may be integrated.

If $\varphi r = mr^2$, (268), becomes,

(269) $d\omega = mdr, \quad \therefore \quad \omega = mr + C,$

which is the spiral of Archimedes.

APPLICATION 23d.

Determine the curve whose polar area varies as a given function of either or both of the polar co-ordinates.

By Proposition VI., (133), we have the polar area, and by the problem, we have the equations,

(270) $\frac{1}{2}\int r^2 d\omega = \phi\omega, \quad \frac{1}{2}\int r^2 d\omega = \varphi r, \quad \frac{1}{2}\int r^2 d\omega = \omega(r,\omega).$

These equations may be integrated for any particular values of the given functions.

APPLICATION 24th.

Determine the curve which cuts the radius vector at a given angle.

By Proposition D, Appendix to Differential Calculus, we have the value of this angle in terms of the tangent, sine, and cosine of the angle. Either of these put equal to a constant, and integrated, gives the curve. If we take the tangent, then putting m for the constant, we have,

$$\frac{rd\omega}{dr} = m, \tag{271}$$

for the differential equation. Integrate this, and we have,

$$\omega = m \log.r + C, \tag{272}$$

the curve required, which is the logarithmic spiral.

APPLICATION 25th.

Determine the curve which cuts the radius vector at an angle whose tangent, or sine, or cosine, varies as a given function of either or both of the polar co-ordinates.

By the same Proposition D, we have the values of these trigonometrical lines, any one of which put equal to a given function of one or both of the polar co-ordinates, furnishes the differential equation, whose integral is the curve required.

If, for example, the tangent varies as the variable angle, we have,

$$\frac{rd\omega}{dr} = m\omega, \tag{273}$$

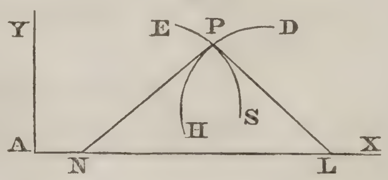
for the differential equation.

$$\therefore \quad \frac{d\omega}{\omega} = m\frac{dr}{r} \quad \therefore \quad \log.\omega = m \log.cr, \text{ or } \omega = (cr)^m. \tag{274}$$

APPLICATION 26th.

Determine the curve which cuts a given family of curves at a given angle.

FIG. 90.



We have already spoken of a given family of curves as a system of curves formed by giving different values to a parameter that enters the equation of a curve. Thus in the equation of the parabola,

(275) $$y^2 = px,$$
if p take different values, (275) expresses different curves, but all of the same family or species, viz: parabolas.

Let the equation of a family of curves be represented by,

(276) $$\phi(x,y,\beta) = o,$$
where β is a parameter that enters the curve. Let DH be one of the curves of this family. Let SE be the curve which cuts DH at a given angle. Draw PN and PL tangent to these curves at the point of intersection P. Put V for the tangent of the given angle NPL, and suppose,

(277) $$\psi(x',y') = o,$$
be the equation of SE, the curve required.

Put p and p' for the differential coefficients of (276) and (277), respectively, then by Trigonometry, we have,

(278) $$\mathrm{V} = \frac{p' - p}{1 + pp'}.$$

The value of p involves the x,y and β of (276). Hence (278) is a function of x,y,β,p', and may be represented by

(279) $$\mathrm{V} = \mathrm{F}\,(x,y,\beta,p').$$

Eliminate β between (279) and (276), and we may represent the result by

(280) $$\mathrm{V} = f\,(x,y,p').$$

But the x and y of (280) appertain to the point of intersection P, and are consequently the same as x',y', the co-ordinates of SE, the curve required. Hence (280) may be written, [putting for p', its equal $dy' \div dx'$],

(281) $$\mathrm{V} = f\left(x',y',\frac{dy'}{dx'}\right).$$

This is the differential equation of the curve required. Its integral is represented by (277). The curve (277) is called *The Trajectory* of the family of curves (276).

If the angle of intersection P be a right angle, (278) becomes,

(282) $$1 + pp' = o,$$
and the trajectory is called *orthogonal*. For the orthogonal trajectory, proceed with (276) and (282) as above directed, for (276) and (278).

We will add several examples of this Proposition.

Ex. 1.—Determine the orthogonal trajectory of the curve,

(a) $$= \beta x^2.$$

Differentiate this, and we have, $\dfrac{dy}{dx} = 2\beta x = p$.

This value of p put into (282), we have,

(b) $$1 + 2\beta x p' = o.$$

Eliminate β between (a) and (b), and we have, [putting x' and y' for x and y],

(c) $$x'dx' + 2y'dy' = o.$$

The integral of this is, $x'^2 + 2y'^2 = \mathrm{C}$, an ellipse.

Ex. 2.—Determine the orthogonal trajectory of the curve,

(d) $$y = \beta x^n.$$

The differential of this is, $p = n\beta x^{n-1}$. This put into (282), we have,

(e) $$1 + n\beta x^{n-1} p' = o.$$

Eliminate β between (d) and (e),

(f) $$\therefore \quad x'dx' + ny'dy' = o, \quad \therefore \quad x'^2 + ny'^2 = \mathrm{C},$$

an ellipse. If $n = 1$, (d) is a family of straight lines, and (f) is a circle.

Ex. 3.—Determine the trajectory which cuts, at a given angle, the family of curves.

(g) $$y = ax.$$

Differentiate this, and we have, $p = a = \dfrac{y}{x}$. This value of p put into (278), we have,

(h) $$\mathrm{V} = \frac{p'x - y}{x + p'y} \quad \therefore \quad (\mathrm{V}x' + y')dx' = (x' - \mathrm{V}y')dy'.$$

This is a homogeneous equation, which may be integrated by Proposition XIII., Case 3. This example is evidently the same as Application 24th, above.

Ex. 4.—Determine the orthogonal trajectory of the family of curves.

(k) $$y^2 = 2rx - x^2,$$

which is a family of circles tangent to the axis of y at the origin.

From (k) we have, for the differential coefficient,

$$(l) \qquad p = \frac{r - x}{y}$$

which put into (282), we have,

$$(m) \qquad y + (r - x)p' = o.$$

Eliminate r between (k) and (m), and we have, [after putting x' and y' for x and y],

$$(n) \qquad 2x'y'dx' + (y'^2 - x'^2)dy' = o,$$

which is a homogeneous equation, and integrated at Proposition XIII., Case 3, from which we have, for the required trajectory, a circle passing through the origin, and tangent to the axis of x.

We will give other Geometrical Applications under other Propositions, preserving the order of the number of the applications.

PROPOSITION XIV.

Determine the integral of differential equations of the first order, exceeding the first degree.

The most general form of a differential equation of two variables, and of any degree n, is

$$(283) \qquad \frac{dy^n}{dx^n} + \mathrm{M}\,\frac{dy^{n-1}}{dx^{n-1}} + \;.\;.\;.\;.\;. + \mathrm{N}\,\frac{dy}{dx} = \mathrm{P},$$

where M, N, and P are functions of x and y. Solve (283) for $\frac{dy}{dx}$, and representing its roots by p, p', p'', &c., we may divide (283) into the several simple equations,

$$(284) \quad \frac{dy}{dx} - p = o, \quad \frac{dy}{dx} - p' = o, \quad \frac{dy}{dx} - p'' = o, \text{ \&c.}$$

The integral of any one of (284) is the primitive of (283), which consequently has n primitives.

Though many of the Applications given under last Proposition involved differential equations of the second degree, yet as the integrations under those Applications were very simple, we did not deem it important to distract the attention of the student by introducing into those Applications the consideration of the differential equations

exceeding the first degree, especially as the results in the Applications given would be, in general, the same. We will now explain the present Proposition by an example or two.

Ex. 1.—Integrate the equation,

$$(a) \qquad dy^2 = m^2dx^2.$$

Here the two roots of the differential coefficient are $+ m$, and $- m$, hence (a) is resolvable into the two equations,

$$(b) \qquad dy = mdx, \qquad \text{and}$$

$$(d) \qquad dy = -mdx.$$

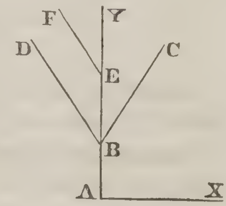
The integrals of these are,

$$(e) \qquad y = mx + \mathrm{C}, \qquad \text{and}$$

$$(f) \qquad y = -mx + \mathrm{C}'.$$

Either of the equations (e) and (f), is a primitive of (a).

FIG. 91.



Equation (e) designates a line BC, whose ordinate at the origin is $\mathrm{AB} = \mathrm{C}$, and (f) designates a line EF, whose ordinate at the origin is $\mathrm{AE} = \mathrm{C}'$. If $\mathrm{C}=\mathrm{C}'$, (e) and (f) designate the two lines BC and BD, having a common ordinate AB at the origin, and making equal angles with the axis of y. Hence we may add the same constant C after the integral of each of the equations (284).

If we multiply (e) and (f) together, we have, making $\mathrm{C} = \mathrm{C}'$,

$$(g) \qquad (y - mx - \mathrm{C})\,(y + mx - \mathrm{C}) = o,$$

which single equation represents the complete integral of (a).

The same thing may be represented by putting (a) in the form,

$$(h) \qquad dy = \pm\, mdx, \quad \therefore \quad y = \pm\, mx + \mathrm{C},$$

which represents two lines BC and BD, equally inclined to the axis of y, at the same point B. In the same manner, if each of the equations (284) be integrated, and the same constant added to each, the product of these integrals is the complete primitive of (283).

Ex. 2.—Integrate the equation,

$$(k) \qquad dy^2 = axdx^2.$$

The two integrals of this appertain to the two branches of the semicubical parabola.

If we could always solve (283) for the differential coefficient, the foregoing process would suffice to integrate equations exceeding the first order. But (283) cannot in general be solved, when it exceeds the fourth degree, and even in those cases where it is resolvable, the radicals are often so complicated that the integral of the partial equations (284) cannot be obtained. The solution and integration of (283) can, however, be effected in the following cases.

Case 1.

If the equation (283) does not contain y, and can be readily solved for x, it can be integrated as follows.

Put p for the differential coefficient $dy \div dx$, and if (283) be solved for x, we may represent the result by

(285) $$x = \mathrm{F}p.$$

Integrate by parts, the equation $dy = pdx$, and we have,

(286) $$y = px - \int xdp.$$

By means of (285), this becomes,

(287) $$y = p\,\mathrm{F}p - \int \mathrm{F}p\,dp.$$

After integrating the last term, of (287), the coefficient p may be eliminated between (285) and (287), and we have the integral required.

Ex.—Integrate $x + mp = n(1 + p^2)^{\frac{1}{2}}$.

This is readily solved for x, and may be integrated as directed in (285) to (287).

If the equation (283) cannot be solved readily for either x or p, put $p = ux$, as in the example,

(l) $$x^3 + p^3 = mpx.$$

Solve this for x, and we have,

$$x = \frac{mu}{1 + u^3}, \quad \text{and} \quad p = \frac{mu^2}{1 + u^3}.$$

Put these values of x and p into the form,

(m) $$y = \int pdx,$$

which is the integral of the assumed equation, $dy = pdx$, and we have,

(n) $$y = m^2 \int \frac{u^2du\,(1 - 2u^3)}{(1 + u^3)^3}.$$

Integrate (n), and u may be eliminated between it and the value of x in (l).

Case 2.

If (283) does not contain x, and can be readily solved for y, represent it by

(288) $$y = \mathrm{F}p.$$

Differentiate this, and we have, $dy = d\mathrm{F}p = pdx$. Solve this for dx, and we have,

(289) $$dx = \frac{d\mathrm{F}p}{p}, \quad \therefore \quad x = \int \frac{d\mathrm{F}p}{p}.$$

Eliminate p between this and $y = \mathrm{F}p$, and we have a relation between y and x, which is the primitive of (288).

Case 3.

If the equation be homogeneous in respect of x and y, the variables may be separated as in the case of homogeneous equations at Proposition XIII., Case 3. For if we assume,

(290) $$y = zx,$$

and substitute into the homogeneous equation, x will disappear from that equation. If the equation can then be solved for z, we have,

(291) $$z = fp, \quad \therefore \quad dz = dfp,$$

and from (290), we have,

(292) $$dy = zdx + xdy = pdx.$$

Eliminate z and dz from this last equation, by means of (291), and we have,

(293) $$\frac{dx}{x} = \frac{dfp}{p - fp}.$$

Eliminate p between the integral of this and the equation, $y = xfp$, [from (290,) (291),] and we have a relation between x and y, which is the integral required.

Ex.—Integrate the equation, $py = nx\,(1 + p^2)^{\frac{1}{2}}$.

Case 4.

If the equation be of the form,

(294) $$y = px + \mathrm{F}p,$$

it is always integrable. Here Fp denotes a function of p, but contains neither x nor y.

Differentiate (294), and we have,

(295) $$dy = pdx + xdp + F'pdp.$$

But since

(296) $$dy = pdx,$$

subtracting, we have,

(297) $$o = (x + F'p)dp.$$

This gives the two conditions,

(298) $$dp = o, \quad \text{and} \quad x + F'p = o.$$

The first of these gives $p = c$. Hence the integral of (294) is,

(299) $$y = cx + Fc.$$

Hence the complete primitive of the form (294) is obtained by making p constant. Again, if we eliminate p between (294) and the second of (298), the resulting equation, which may be represented by

(300) $$F(x,y) = o,$$

will be an integral of (294), but will not contain any arbitrary constant. This result is called a *singular solution* of (294). We will revert to these solutions hereafter. The form (294) is known as *Clairaut's Form*. The second of (298) might also be integrated by putting for p, its value, $\frac{dy}{dx}$.

Case 5.

If the equation be of the form,

(301) $$y = Px + P',$$

where P and P$'$ are functions of p, we first differentiate (301), and we have,

(302) $$dy = Pdx + xdP + dP'.$$

Since we have,

(303) $$dy = pdx,$$

by subtraction we have,

(304) $$o = (P - p)\,dx + xdP + dP'.$$

From this we get,

(305) $$dx + \frac{xdP}{P-p} = -\frac{dP'}{P-p},$$
which is the same form as (205).

We will give a few Geometrical Applications involving the preceding cases.

APPLICATION 27th.

Determine the curve whose tangent makes with the axis of ordinates an angle whose cosine varies as a given function of the co-ordinates of the point of tangency.

Since the differential of the arc is $(dx^2 + dy^2)^{\frac{1}{2}}$, the cosine of the proposed angle is, $\frac{dy}{(dx^2 + dy)^{\frac{1}{2}}}$, and by the problem, we have,

(306) $$\frac{dy}{(dx^2 + dy^2)^{\frac{1}{2}}} = \varphi(x,y).$$

If $\varphi(x,y) = \frac{nx}{y}$, (306) becomes,

(307) $$ydx = nx\,(dx^2 + dy^2)^{\frac{1}{2}},$$
which being homogeneous, may be treated by Case 3, Prop. XIV.

APPLICATION 28th.

Determine the curve in which the rectangle of the abscissa, and abscissa diminished by a given line, equals the rectangle of the arc and that line.

By the problem, we have, putting n for the given line,

(308) $$x\,(x - n) = n\int (dx^2 + dy^2)^{\frac{1}{2}}.$$

Differentiate this, to dispose of the sign of integration, and we have,

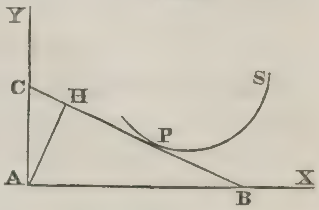
(309) $$2xdx - ndx = n\,(dx^2 + dy^2)^{\frac{1}{2}},$$
or dividing by dx, we have,

(310) $$2x - n = n\,(1 + p^2)^{\frac{1}{2}},$$
which may be treated by Case 1, Proposition XIV., and the curve determined.

APPLICATION 29th.

Determine the curve whose tangent cuts off, with the co-ordinate axes, a triangle of given area.

FIG. 92.



Let the equation of the tangent line CB be,

(311) $y - y' = p\,(x - x')$.

The distances from the origin to where this line cuts the axes of x and y, are found by putting first, $x = o$, and then $y = o$, in (311), from which we have,

(A) $y' - px' = \text{AC}$, and $-\dfrac{(y' - px')}{p} = \text{AB}$.

Putting a^2 for the area of the triangle CAB, we have,

(312) $$(y' - px')^2 = -2pa^2.$$

The negative sign may be changed to positive, as it only affects the position of the area. Hence (312) becomes, omitting the accents, extracting the root, and transposing,

(313) $$y = px + a\sqrt{2p}.$$

This is of the form (294). Hence its integral is,

(314) $$y = cx + a\sqrt{2c},$$

which is the equation of a straight line If (313) be differentiated, the particular form of (298) is,

(315) $$x + \frac{a}{\sqrt{2p}} = o.$$

Eliminating p between (313) and (315), we have, for the *singular solution*,

(316) $$xy = -\frac{a^2}{2},$$

a hyperbola, as in Proposition XXXVII., Example A, Differential Calculus. This would be the curve PS.

If the second side of (312) be treated with the negative sign, we have, instead of (313) and (315),

(317) $$y = px + a\sqrt{-2p}, \quad \text{and} \quad x + \frac{a}{\sqrt{-2p}} = o.$$

If p be eliminated between these two, we have,

(318) $$xy = \frac{a^2}{2}, \qquad \text{as before.}$$

APPLICATION 30th.

Determine the curve whose tangent between the axes is always of the same length.

Put m for the given length CB, figure 92, and using the lengths (A), Application 29th, we have,

(319) $$(y - px)^2 + \frac{(y - px)^2}{p^2} = m^2.$$

From this we get,

(320) $$y = px + \frac{pm}{(p^2 + 1)^{\frac{1}{2}}},$$

which is of Clairaut's form (294). Hence put C for p in (320), and we have its complete primitive. Differentiate (320), and the particular form of the second of (298) is,

(321) $$o = x + \frac{m}{(p^2 + 1)^{\frac{3}{2}}}.$$

If p be eliminated between (321) and (320), we have for the singular solution,

(322) $$y^{\frac{2}{3}} + x^{\frac{2}{3}} = m^{\frac{2}{3}}.$$

If we put in (321), for p, its value $\frac{dy}{dx}$, and then integrate (321), we find,

(323) $$y = -(m^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}} + \text{C}.$$

To determine C, observe that when $y = o$, we have $x = m$. Hence $\text{C} = o$ and (323) becomes (322).

APPLICATION 31st.

Determine the curve such that a perpendicular drawn from a given point to its tangent, is of a given length.

Suppose the given point be the origin, and m the given length AH, [fig. 92], of the perpendicular. For the tangent CB we have the equation,

(324) $$y - y' = p\,(x - x').$$

For the equation of the perpendicular AH, on (324), from the origin, we have,

(325) $$py = -x.$$

For the length of the perpendicular AH, we have,

(326) $$m^2 = x^2 + y^2.$$

Eliminate x and y from these three equations, and we have,

(327) $$y' = px' + m\,(p^2 + 1)^{\frac{1}{2}}.$$

This being of Clairaut's form, (294), its integral is,

(328) $$y' = cx' + m\,(c^2 + 1)^{\frac{1}{2}}.$$

The singular solution of (327) is,

(329) $$x'^2 + y'^2 = m^2,$$

which is the equation of a circle.

PROPOSITION XV.

Determine the singular solution of a differential equation.

We have given singular solutions of the last three Applications, and from the mode of determining them, it is obvious that the complete primitive being the equation of a line, [straight or curved], the singular solution is the locus of the intersection of this line with its consecutive line. For equation (315), for example, is obviously the same as would be obtained by differentiating (313) for x and y constant, and p variable, and the result of eliminating p between (313) and (315) is obviously the same as the result of eliminating c between (314) and the differential of (314), for c variable. Hence the determination of *singular solutions* is resolved into the determination of the locus of intersection of a line with its consecutive line, a subject fully investigated in the Differential Calculus, Proposition XXXVII.

To obtain the singular solution of a differential equation, there-

fore, we first obtain its complete primitive, and eliminate the arbitrary constant from this complete primitive, by the principle of consecutive lines. If the arbitrary constant disappears in differentiating the primitive for this constant variable, the equation does not admit of a singular solution.

By eliminating c between (328) and its differential for c variable, we get the singular solution (329). Consequently the curve given by the singular solution touches all the lines contained in the complete primitive.

The singular solution of Application 31st is the same as would be obtained by the principle of consecutive lines, if the Application had been in these words: *A line is drawn at a given distance from a given point; determine the locus of intersection of this line with its consecutive line.* Equation (328) is the equation of a straight line in terms of the perpendicular on it from the origin, and of the angle this perpendicular makes with the axis of x.

Ex.—Determine the singular solution of the equation,

$$(a) \qquad xdy - ydx = (x^2 + y^2)^{\frac{1}{2}}dx.$$

The complete integral of this is, [Proposition XIII., Case 3, Ex.2],

$$(b) \qquad x^2 = 2cy + C^2.$$

Hence the singular solution is,

$$(c) \qquad x^2 + y^2 = o, \qquad \text{a point.}$$

For the mode of obtaining the singular solution without first integrating the differential equation, see Proposition (E), post.

PROPOSITION XVI.

Determine the integral of a differential equation of the second order, containing two variables.

The most general form in which such an equation can occur, is,

$$(330) \qquad \varphi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = o,$$

which contains the first and second differential coefficients, and both the variables.

We will not attempt the integration of (330) in its most general form, but will examine several cases of it.

Case 1.

If (330) contain neither y nor the first differential coefficient, it becomes,

(331) $$\varphi\left(x, \frac{d^2y}{dx^2}\right) = o.$$

This solved for the second differential coefficient, gives that coefficient as a function of x, which denote by πx, and we have,

(332) $$\frac{d^2y}{dx^2} = \pi x.$$

This may be integrated as at (106).

Case 2.

If (330) contains neither x nor the first differential coefficient, it becomes,

(333) $$\varphi\left(y, \frac{d^2y}{dx^2}\right) = o.$$

This solved for the second differential coefficient, we have,

(334) $$\frac{d^2y}{dx^2} = \pi y.$$

Assume $\frac{dy}{dx} = p$, then by differentiating, we have,

(335) $$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{pdp}{dy}.$$

Equating (334) and (335), we have,

(336) $$pdp = \pi ydy, \quad \therefore \quad p^2 = 2\int \pi ydy = y' + \mathrm{C},$$

where y' is a function of y, and is put for the integral of $2\int \pi ydy$. Restore the value of p in (336), and we have,

(337) $$\frac{dy}{dx} = (y' + \mathrm{C})^{\frac{1}{2}}, \quad \therefore \quad x = \int \frac{dy}{(y' + \mathrm{C})^{\frac{1}{2}}}$$

the integral required.

Case 3.

If (330) contain neither x nor y, it becomes,

(338) $$\varphi\left(\frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = o.$$

Assume $\frac{dy}{dx} = p$, $\therefore$ $\frac{d^2y}{dx^2} = \frac{dp}{dx}$. These values put into (338), we have,

(339) $$\phi\left(p, \frac{dp}{dx}\right) = o,$$

an equation containing p and x, of the first order. Solve it for dx, and we have,

(340) $$dx = \int pdp, \quad \therefore$$

(341) $$x = \int' p + C.$$

But since $dy = pdx$, we have,

(342) $$dy = p\int pdp, \quad \therefore$$

(343) $$y = \int'' p + C'.$$

Eliminate p between (341) and (343), and we have,

(344) $$F\,(x,y,C,C') = o,$$

the complete primitive of (338).

Case 4.

If the equation (330) does not contain y, it becomes,

(345) $$\phi\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = o.$$

Assume $\frac{dy}{dx} = p$, $\therefore$ $\frac{d^2y}{dx^2} = \frac{dp}{dx}$. These values put into (345), it becomes,

(346) $$\phi\left(x,p, \frac{dp}{dx}\right) = o,$$

which is an equation of the first order in p and x, and may be integrated by Proposition XIV.

Suppose its integral determined, and of the form, $F\,(x,p,C,) = o$.

In this put for p, its value, and we have again a differential equation of the first order, which may be treated by Proposition XIV. The result will be,

(348) $$f\,(x,y,C,C') = o,$$

the complete primitive.

Case 5.

If (330) does not contain x it becomes,

(349) $$\phi\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = o.$$

Since $\dfrac{dy}{dx} = p$, and $\dfrac{d^2y}{dx^2} = \dfrac{pdp}{dy}$, (349) becomes,

(350) $$\varphi\left(y, p, \frac{pdp}{dy}\right) = o,$$

a differential equation of the first order in y and p, which may be treated by Proposition XIV.

Case 6.

If (330) be homogeneous in respect of the variables and differential coefficients it may be reduced to an equation of the first order.

By the equation being homogeneous in respect of the differential coefficients, is understood that the degree of homogeneity of $\dfrac{dy}{dx}$ is zero, and of $\dfrac{d^2y}{dx^2}$ is -1. Thus the equation,

(351) $$x^2 \frac{d^2y}{dx^2} + x + y \frac{dy}{dx} = o,$$

is homogeneous, the degree of homogeneity being unity. Assume,

(352) $$\frac{dy}{dx} = p, \quad y = ux, \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{z}{x}.$$

These put into (330) will render (330), when homogeneous in the n^{th} degree, divisible by x^n, and dividing by x^n, (330) will contain only the three quantities, p,u,z, and may be written,

(353) $$\phi(p,u,z) = o.$$

Differentiate $y = ux$, and we have, $dy = udx + xdu$; from this, and $dy = pdx$, we get

(354) $$\frac{dx}{x} = \frac{du}{p - u}.$$

But from the first of (352) we have,

$$\frac{d^2y}{dx^2} = \frac{dp}{dx},$$

which equated with the last of (352), gives,

$$\frac{dp}{dx} = \frac{z}{x}, \quad \text{or}, \quad \frac{dx}{x} = \frac{dp}{z}.$$

This equated with (354), we have,

(355) $$\frac{dp}{z} = \frac{du}{p - u}.$$

Eliminate z between (353) and (355), and the result contains two variables, and may be represented by

(356) $$\varpi(p,u,dp,\ du) = o.$$

This being of the first order, may be integrated by Proposition XIV. Then by means of this integral, p may be eliminated from (354), and that equation integrated, we have,

(357) $$\log.x = fu.$$

Eliminate u between this and $y = ux$, and we have the relation between x and y.

We will add a few Geometrical Applications involving these cases.

APPLICATION 32d.

Determine the curve whose radius of curvature is constant.

By equation (199), Differential Calculus, we have the length of the radius of curvature of any curve, which if we put equal to a constant m, we have,

(358) $$\pm\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}} \div \frac{d^2y}{dx^2} = m.$$

This containing neither x nor y, but the first and second differential coefficients, is a particular example under Case 3. Hence taking (358) with the negative sign, it becomes, in terms of p and x,

(359) $$dx = -\frac{mdp}{(1 + p^2)^{\frac{3}{2}}}.$$

Integrate this, and we have,

$$x = \frac{mp}{(1 + p^2)^{\frac{1}{2}}} + \text{C}.$$ Since,

(360) $$dy = pdx, \quad \therefore \quad y = \frac{m}{(1 + p^2)^{\frac{1}{2}}} + \text{C}'.$$

Eliminate p between (359) and (360), and we have for the primitive,

(361) $$(\text{C} - x)^2 + (\text{C}' - y)^2 = m^2,$$

a circle, the radius of curvature being the radius of the circle.

APPLICATION 33d.

Determine the curve whose radius of curvature varies as the square of the abscissa.

By the problem, we have,

(362) $$-\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}} \div \frac{d^2y}{dx^2} = \frac{x^2}{m},$$

for the differential equation, which is a particular example of Case 4.

Substituting, as in Case 4, we have,

(363) $$-(1+p^2)^{-\frac{3}{2}}dp = mx^{-2}dx,$$

which integrated, [by (78),] gives,

(364) $$\frac{p}{(1+p^2)^{\frac{1}{2}}} = \frac{m}{x} + \text{C}.$$

This may be again integrated by Proposition XIV., Case 1, and the curve determined.

APPLICATION 34th.

Determine the curve whose radius of curvature equals the normal.

Putting the length of the normal equal to the radius of curvature, we have,

(365) $$y\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = -\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}} \div \frac{d^2y}{dx^2}.$$

This is a particular example of Case 5. Changing this to p and y, we have,

(366 $$\frac{dy}{y} = -\frac{pdp}{p^2+1}.$$

The integral of this is,

$$y^2 = \frac{\text{C}}{p^2+1}.$$

Restoring the value of p, and integrating again, we get,

(367) $$(x-\text{C}')^2 + y^2 = \text{C}, \text{ a circle.}$$

If we take the second side of (365) positive, we get, instead of (366),

(368) $$y^2 = C\,(p^2 + 1),$$

and restoring the value of p, we get for the next integral, a logarithmic equation. Now we know, Differential Calculus, that the radius of curvature with a negative sign, gives a curve concave towards the axis of abscissas, and with a positive sign, a curve convex towards the same axis. Hence the logarithmic curve given by integrating (368) is convex towards the axis of x. This curve has many properties analogous to those of the circle.

APPLICATION 35th.

Determine the curve whose radius of curvature is a given multiple of the normal.

Let n be the given multiple. Then putting n times the first side of (365) equal the second side, and proceeding as in Application 34, we get for the final integral,

(369) $$x = \int y^{\frac{1}{n}} dy\,(c - y^{\frac{2}{n}})^{-\frac{1}{2}}.$$

This fulfils the condition of integrability at (65), if n be an odd number, and the condition of integrability at (66a), if n be an even number. Hence the curve is always determinable for integer values of n.

APPLICATION 36th.

Determine the curve whose radius of curvature is a given multiple of the distance from a given point to the point of osculation.

Let the origin be the given point, and n the given multiple, then by the problem, we have,

(370) $$n\,(x^2 + y^2)^{\frac{1}{2}} = -\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}} \div \frac{d^2y}{dx^2}.$$

This is a particular example of Case 6, last Proposition, and proceeding with it as directed in Case 6, we get for the particular form of (356), the equation,

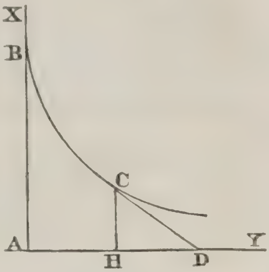
(371) $$(1 + p^2)^{\frac{3}{2}} du = n\,(p - u)\,(1 + u^2)^{\frac{1}{2}} dp,$$

an equation integrated by Euler, by means of circular arcs. [Vide La Croix, vol. II., 307].

APPLICATION 37th.

A body D moves with a given velocity along a given straight line, a body C moves with a given velocity directly towards D ; determine the locus of C.

FIG. 93



Let D move on the axis of y, and since the velocities are given, their ratio n is given.

When D is at A, let C be at B, on the axis of x, and let the points C and D be their position at any other time.

Put AH $= y$, HC $= x$, BC $= z$. Then the subtangent is,

$$\text{HD} = -\frac{xdy}{dx},$$

this increased by AH, or x, is the distance gone by D while C goes the distance BC $= z$.

The ratio n of these distances being given, we have,

$$(372) \qquad \left(y - x\frac{dy}{dx}\right) = nz,$$

or putting for z its value in (122),

$$(373) \qquad (y - px) = n\int(1 + p^2)^{\frac{1}{2}}dx.$$

This differentiated to dispose of the sign of integration, we have,

$$(374) \qquad -n\frac{dx}{x} = \frac{dp}{(1 + p^2)^{\frac{1}{2}}},$$

which comes under Case 4. By (63) the integral of this is,

$$(375) \qquad \therefore \quad (Cx)^n = (1 + p^2)^{\frac{1}{2}} - p.$$

To determine C, we observe that when $x =$ AB $= a$, we have, $p = o$. These values put into (375), we have, $C = \dfrac{1}{a}$.

This value of C put into (375), that equation becomes, after restoring the value of p, and integrating,

$$(376) \qquad 2y = \frac{x^{n+1}}{a^n(n+1)} + \frac{a^n}{(n+1)x^{n-1}} + C.$$

From this C may be eliminated by the consideration that when $y = o$, $x = a$, and the curve is completely determined.

This is *The Curve of Pursuit* of the French Mathematicians.

PROPOSITION XVII.

Determine the integral of differential equations of two variables exceeding the second order.

We shall attempt this in only a few cases.

Case 1.

If the equation be of the form,

$$F\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}\right) = o. \tag{377}$$

Assume,

$$\frac{d^{n-1}y}{dx^{n-1}} = u, \quad \therefore \quad \frac{d^ny}{dx^n} = \frac{du}{dx}. \tag{378}$$

These values put into (377), we have,

$$F\left(\frac{du}{dx}, u\right) = o, \tag{379}$$

which is of the first order in respect of u and x. This being integrated, we may represent the result by $u = fx$, and this value of u put into the first of (378), we get a form which may be integrated, and gives a relation between y and x.

Case 2.

If the equation be of the form,

$$F\left(\frac{d^ny}{dx^n}, \frac{d^{n-2}y}{dx^{n-2}}\right) = o. \tag{380}$$

Assume,

$$\frac{d^{n-2}y}{dx^{n-2}} = u, \quad \therefore \quad \frac{d^ny}{dx^n} = \frac{d^2u}{dx^2}. \tag{381}$$

These values put into (380), we have,

$$F\left(\frac{d^2u}{dx^2}, u\right) = o, \tag{382}$$

which is of the second order, and may be integrated by Case 2, Proposition XVI. If the integral of (382) be $u = fx$, this value of u put into the first of (381), we may integrate (381), and find the relation between y and x.

Case 3.

Suppose the equation be of the form,

(383) $$d^n y + Pd^{n-1}ydx + Qd^{n-2}ydx^2 + \dots . Sydx^n = o,$$

where P, Q, &c., do not contain y, but may contain x. This form of the equation contains y only in the first power, and is therefore called a linear equation. Suppose, for illustration, (383) be of the third order,

(384) $$d^3y + Pd^2ydx + Qdydx^2 + Sydx^3 = o.$$

Assume,

(A) $$y = e^{\int udx}, \therefore dy = e^{\int udx}udx, \; d^2y = e^{\int udx}(u^2dx^2 + dudx).$$

(B) $$d^3y = e^{\int udx}(u^3dx^3 + 3ududx^3 + d^2udx).$$

These values put into (384), it becomes, after dividing by dx, and by the exponential,

(385) $$d^2u + (3u + P)dudx + (u^3 + Pu^2 + Qu + S)dx^2 = o,$$

an equation of the second order in u and x.

If P, Q, and S be constant, we may take u constant. For if u be constant, $du = o$, $d^2u = o$, and (385) becomes,

(386) $$u^3 + Pu^2 + Qu + S = o,$$

from which u may be determined.

Let u', u'', u''', be the three roots of (386). Then designating the values of y in A, corresponding to these three roots by y', y'', y''', we have, from the first of (A),

(387) $$y' = e^{u'x+c'}, \quad y'' = e^{u''x+c''}, \quad y''' = e^{u'''x+c'''},$$

or putting C′, C″, C‴, for $e^{c'}$, $e^{c''}$, $e^{c'''}$, (387) becomes,

(388) $$y' = C'e^{u'x}, \quad y'' = C''e^{u''x}, \quad y''' = C'''e^{u'''x}.$$

Each of these values satisfies (384), and are particular cases of the complete primitive of that equation. As the complete primitive contains each of the values (388), it may be represented by the sum of (388), viz:

(389) $$y = C'e^{u'x} + C''e^{u''x} + C'''e^{u'''x}.$$

For if we substitute into (384) the values of the differentials of (388), that equation will be satisfied.

In the same manner may the equation (383) be integrated, when P, Q, &c., are constants.

If P, Q, &c., are functions of x, then if we knew three values of y, viz: $C'y'$, $C''y''$, $C'''y'''$, which would satisfy (384), the complete primitive would be their sum, viz:

(390) $$y = C'y' + C''y'' + C'''y'''.$$

For if the several differentials of (390) be put into (384), then by putting (384) equal to M, and by putting M′, M″, M‴, for the corresponding values of (384) in y', y'', y''', we have,

(391) $$C'M' + C''M'' + C'''M''' = o,$$

a result in which M′, M″, and M‴, are separately zero, since each of the values $C'y'$, $C''y''$, $C'''y'''$, satisfies (384).

For more particular details on the integration of the form (383), consult La Croix, vol. II., p. 313—337.

PROPOSITION XVIII.

Determine the integral of a total differential equation of three variables, when the first side of the equation contains one variable, and the other side the other two.

In the Differential Calculus, [Proposition LXIX., (377),], we found the total differential of the equation,

(392) $$z = \varphi(x,y).$$

to be,

(393) $$dz = \frac{dzdx}{dx} + \frac{dzdy}{dy},$$

in which each term on the second side is the partial differential of (392).

Case 1.

Suppose now we have to integrate the equation,

(394) $$dz = Pdx + Qdy,$$

in which P and Q are any functions of x and y.

If (394) has been derived immediately from differentiating an

equation such as (392), it is called an *Exact Differential.* The first examination of (394) is to ascertain if it be an exact differential. If it be so, then P and Q are partial differential coefficients, and are equal to the partial differential coefficients of (393), that is,

(395) $$P = \frac{dz}{dx}, \qquad Q = \frac{dz}{dy}.$$

Differentiate the first of these relatively to y, and the second relatively to x, and we have,

(396) $$\frac{dP}{dy} = \frac{d^2z}{dxdy}, \qquad \frac{dQ}{dx} = \frac{d^2z}{dydx}.$$

Since, [by Differential Calculus, Proposition LXVIII., Cor. —],

$$\frac{d^2z}{dxdy} = \frac{d^2z}{dydx}, \text{ we have,}$$

(397) $$\frac{dP}{dy} = \frac{dQ}{dx}.$$

When this condition exists, (394) is an exact differential.

In the same manner, if we have an equation of four variables,

(398) $$dz = Pdx + Qdy + Rdu,$$

we have, in case it be an exact differential, the conditions,

(399) $$\frac{dP}{dy} = \frac{dQ}{dx}, \quad \frac{dP}{du} = \frac{dR}{dx}, \text{ and } \frac{dQ}{dz} = \frac{dR}{dy}.$$

This we will exhibit at large hereafter.

Ex.—Suppose we have, as a particular case of (394), the equation,

(*a*) $dz = (6xy - y^2)dx + (3x^2 - 2xy)dy.$ Here,

(*b*) $P = 6xy - y^2,$ and $Q = 3x^2 - 2xy.$

Then differentiating, we have,

(*c*) $$\frac{dP}{dy} = 6x - 2y, \quad \text{and} \quad \frac{dQ}{dx} = 6x - 2y.$$

These being equal, the condition (397) is fulfilled, and (*a*) is an exact differential.

In order to integrate (394), when it is an exact differential of an equation (392), we know that the first term of (394) has been obtained by differentiating (392) for y constant. Hence if we integrate the first term, Pdx, supposing y to be constant, we will have, after the proper correction, the primitive (392). Hence the primitive of (394) is,

(400) $$z = \int \mathrm{P}dx + \mathrm{C}.$$

But since y was made constant in differentiating for the first term of (394), C may contain y, and we may write (400) in the form,

(401) $$z = \int \mathrm{P}dx + y',$$

where y' is put for the function of y and constants that disappeared in differentiating (392) for y constant.

To determine y', differentiate (401) for y as the variable, and we have,

(402) $$\frac{dz}{dy} = \frac{d\int \mathrm{P}dx}{dy} + \frac{dy'}{dy} = \mathrm{Q},$$

from which we have,

(403) $$\frac{dy'}{dy} = \mathrm{Q} - \frac{d\int \mathrm{P}dx}{dy}, \quad \therefore\ y' = \int \left(\mathrm{Q} - \frac{d\int \mathrm{P}dx}{dy}\right)dy.$$

This value put into (401), we have, for the complete integral of (394),

(404) $$z = \int \mathrm{P}dx + \int \left(\mathrm{Q} - \frac{d\int \mathrm{P}dx}{dy}\right)dy.$$

If we had taken the last term, $\mathrm{Q}dy$, of (394) and integrated, we would have obtained a form similar to (404), for the integral.

To apply this process to example (a) above, we have, by integrating the first term of (a) for y constant,

(d) $$z = 3x^2y - y^2x + y'.$$

This differentiated for y variable, is,

(e) $$\frac{dz}{dy} = 3x^2 - 2xy + \frac{dy'}{dy} = \mathrm{Q}.$$ But by (b),

(f) $$3x^2 - 2xy = \mathrm{Q}.$$ Hence from (403),

(g) $$\frac{dy'}{dy} = o, \quad \text{or} \quad y' = \mathrm{C}.$$ Hence (d) is,

(h) $$z = 3x^2y - y^2x + \mathrm{C},$$

the complete primitive of (a).

Ex. 2. Integrate the equation,

(k) $$dz = ydx + (x + a)dy.$$

Here $\mathrm{P} = y$, and $\mathrm{Q} = x + a$, and

$$\frac{d\mathrm{P}}{dy} = 1, \quad \text{and} \quad \frac{d\mathrm{Q}}{dx} = 1.$$

Hence (k) is an exact differential. Integrate it for y constant, and we have,

(l) $$z = yx + y'.$$

Differentiate (l) for x constant, and we have,

(m) $$\frac{dz}{dy} = x + \frac{dy'}{dy} = \mathrm{Q} = x + a, \ \therefore \ dy' = ay, \ \therefore \ y' = ay,$$

and (l) becomes,

(n) $$z = yx + ay.$$

Ex. 3. Integrate the equations,

$$dz = 2xy - (x^2 + 2my)dy.$$

Ans. $z = x^2y - my^2.$

$$dz = \frac{ydx - xdy}{x^2 + y^2}.$$

Ans. $z = \tan.^{-1}\frac{x}{y} + \mathrm{C}.$

Case 2.

If the second side of (394) be homogeneous as well as exact, the integral may be obtained in a very simple manner. Let

(405) $$z = \phi(x,y),$$

be the primitive of the homogeneous equation,

(406) $$dz = \mathrm{P}dx + \mathrm{Q}dy.$$

Let n be the degree of homogeneity of (405), then $n - 1$ is the degree of homogeneity of P and Q, in (406).

Assume $y = rx$. This value of y put into (405), it becomes,

(407) $$z = \mathrm{R}x^n,$$

where R is put for a function of r. These values of y and z put into (406), it becomes, [putting P′ and Q′ for what P and Q then become,]

(408) $$d\,(\mathrm{R}x^n) = \mathrm{P}'dx + \mathrm{Q}'d\,(rx),$$

or performing the differentiations denoted in (408), we have,

(409) $$n\mathrm{R}x^{n-1}dx + x^n d\mathrm{R} = (\mathrm{P}' + \mathrm{Q}'r)dx + \mathrm{Q}'xdr.$$

The first term on each side of this is the differential of (405) relatively to x, and since these partial differentials must be equal, we have,

(410) $$n\mathrm{R}x^{n-1} = \mathrm{P}' + \mathrm{Q}'r.$$

Restore in this the value of r from $y = rx$, and since P′ and Q′ then return to P and Q, (410) becomes,

$$(411) \qquad nRx^{n-1} = P + Q\frac{y}{x},$$

or eliminating R by (407), we have,

$$(412) \qquad z = \frac{Px + Qy}{n},$$

the primitive of (406). Hence to integrate a homogeneous exact differential equation, we need only change dx and dy to x and y, and divide by the degree of homogeneity increased by unity.

Ex.—Integrate the equation,

$$(a) \qquad dz = 3x^2ydx + (x^3 - 4y^3)dy.$$

This being both exact and homogeneous, the degree of homogeneity being 3, its integral is,

$$(b) \qquad z = \frac{3x^3y + (x^3 - 4y^3)y}{4} = x^3y - y^4.$$

Ex. 2. Integrate the equation,

$$(c) \qquad dz = (\tfrac{3}{2}x^2y^2 + y^4)dy + (y^3x - x^4)dx,$$

$$z = \frac{x^2y^3}{2} - \frac{x^5}{5} + \frac{y^5}{5}.$$

The constant should of course be added to complete the integral.

The principle of this case is applicable in the same manner to an exact homogeneous equation of any number of variables. For example, if we have,

$$(413) \qquad dz = Pdx + Qdy + Rdu,$$

by an analogous procedure, we would get, [$n - 1$ being the degree of homogeneity of (413),] for the integral,

$$(414) \qquad z = \frac{Px + Qy + Ru}{n}.$$

PROPOSITION XIX.

Determine the factor which will render a differential equation exact.

Take the equation of two variables.

(415) $$Pdx + Qdy = o,$$

P and Q being functions of x and y.

If (415) be an exact differential, it must fulfil the condition (397), and may be integrated by Proposition XVIII., without separating the variables. It may have happened, however, that a factor containing x or y, or both of them, has disappeared from (415), in consequence of which (415) is not an exact differential. If this factor could be found, and (415) multiplied by it, the equation would then be integrable as an exact differential. Suppose u be the factor that is required to render (415) an exact differential, u being a constant or a function of x or y, or of both of them. Multiply (415) by this factor, and we have,

(416) $$Pudx + Qudy = o.$$

Since this by hypothesis is an exact differential, the condition (397) is fulfilled, and we have,

(417) $$\frac{dPu}{dy} = \frac{dQu}{dx},$$

or performing the differentiation, we have,

(418) $$\frac{Pdu}{dy} - \frac{Qdu}{dx} + \left(\frac{dP}{dy} - \frac{dQ}{dx}\right) u = o.$$

If this equation were integrated, we would have the value of u required. But (418) is more difficult to integrate in its general form than (415), and consequently the factor cannot always be determined. There are, however, two cases when (418) may be integrated, and the factor determined.

Case 1.

Let the factor u be a function of x only.

In this case, $\frac{du}{dy} = o$, and (418) becomes,

(419) $$\left(\frac{dP}{dy} - \frac{dQ}{dx}\right) \frac{dx}{Q} = \frac{du}{u}.$$

But since u is by hypothesis a function of x, the first side of (419) cannot contain y. Hence we may assume,

(420) $$\frac{1}{Q} \left(\frac{dP}{dy} - \frac{dQ}{dx}\right) = fx,$$

and (419) becomes,

(421) $\dfrac{du}{u} = fxdx, \quad \therefore \quad \log.u = \int fxdx = x', \quad \therefore \quad u = e^{x'}.$

Then the equation (416) multiplied by $e^{x'}$, is an exact differential.

Ex. 1. Integrate the equation,

(*a*) $$ydx - xdy = o.$$

Here $P = y$, $Q = -x$, and (420) becomes,

(*b*) $$\frac{1}{-x}(1 + 1) = \frac{2}{-x},$$

and (421) gives,

(*c*) $\dfrac{du}{u} = -\dfrac{2dx}{x}, \quad \therefore \log.u = -2\log.x + \log.c, \quad \therefore \quad u = \dfrac{C}{x^2}.$

Hence (*a*) becomes,

$$\frac{C(ydx - xdy)}{x^2} = o, \qquad \text{whose integral is,} \qquad \frac{Cy}{x} = C'.$$

Ex. 2. Integrate the equation,

$$xdy + (b - 2y)dx = o.$$

The factor is, $u = \dfrac{C}{x^3}.$

In like manner, if the factor were a function of y, its value may be obtained.

Case 2.

Let the equation (415) be homogeneous.

Let u be the factor required, then (415) becomes,

(422) $$Pudx + Qudy = o = dz.$$

This equation being, by supposition, homogeneous, its integral is, by Case 2, last Proposition, [putting $n - 1$ for the degree of homogeneity of (422),]

(423) $$\frac{Pux + Quy}{n} = z.$$

Divide (422) by (423), and we have,

(424) $$\frac{Pudx + Qudy}{Pux + Quy} = \frac{dz}{nz}.$$

The second side of this being an exact differential, the first side must also be an exact differential. Consequently (424) shows that the factor necessary to render (415) a complete integral, is,

(425)
$$u = \frac{1}{Px + Qy}.$$

The same principle could be extended to a homogeneous equation of any number of variables. For if we had an equation as,

(426)
$$Pdx + Qdy + Rdz = o,$$

the factor required is,

(427)
$$u = \frac{1}{Px + Qy + Rz}.$$

Ex.—Integrate the equation,

(a)
$$2xydy - (y^2 - x^2)dx = o.$$

This being homogeneous, the factor (425) is,

(b)
$$u = \frac{1}{xy^2 + x^3}.$$

Equation (a) being multiplied by this factor, may be readily put in the form,

(c)
$$\frac{2ydy + 2xdx}{y^2 + x^2} - \frac{(y^2 + x^2)}{(y^2 + x^2)} \cdot \frac{dx}{x} = o.$$

The integral of this is,

(d)
$$\log.(x^2 + y^2) - \log.cx = o, \quad \therefore \quad x^2 + y^2 - cx = o.$$

Besides these, an indefinite number of factors may be found, which will render integrable the equation (415). For if z be the integral of that equation, and u a factor which renders it exact,

(428)
$$dz = Pudx + Qudy.$$

Multiply this by fz, and we have,

(429)
$$fzdz = Pufzdx + Qufzdy.$$

The form of fz being arbitrary, it may be taken of any value, as z^3, then z^3dz, the first side of (429), being an exact differential, the equation

(430)
$$z^3dz = z^3u\,(Pdx + Qdy),$$

is also an exact differential, and the factor z^3u renders (415) exact.

The process of integration by first determining a factor, is, however, in general complicated, and cannot indeed be practically applied except in cases where the integral may be readily obtained by other processes.

PROPOSITION XX.

Determine the integral of a differential equation of three variables, when the variables enter it in any manner.

Let the equation be,

$$(431) \qquad Pdx + Qdy + Rdz = o,$$

where P, Q, R are each functions of x, y, and z. It is evident that the variables in (431) cannot in general be separated so as to reduce (431) to the form (394). But if the complete primitive of (431) be,

$$(432) \qquad \varphi(x,y,z) = o = u,$$

then since the differential of this is,

$$(433) \qquad \frac{dudx}{dx} + \frac{dudy}{dy} + \frac{dudz}{dz} = o,$$

we have, by comparing (431) and (433), the conditions,

$$(434) \qquad P = \frac{du}{dx},\ Q = \frac{du}{dy},\ R = \frac{du}{dz}.$$

Combining these two and two, as in (395) to (397), we have the conditions,

$$(435) \qquad \frac{dP}{dy} = \frac{dQ}{dx},\ \frac{dP}{dz} = \frac{dR}{dx},\ \frac{dQ}{dz} = \frac{dR}{dy}.$$

When these conditions exist, (431) is an exact differential.

Suppose we have the equation,

$$(a) \qquad 2zxdx + z^2dy + (x^2 + 2zy)dz = o.$$

Here $P = 2zx$, $Q = z^2$, $R = x^2 + 2zy$, and the conditions (435) are fulfilled, and equation (a) is an exact differential.

If (431) does not fulfil the conditions (435), let us suppose (431) to become an exact differential, by being multiplied by some factor u. Suppose that

$$(436) \qquad Pudx + Qudy + Rudz = o,$$

be an exact differential. Then the conditions (435) are,

$$(437) \qquad \frac{duP}{dy} = \frac{duQ}{dx}\ \frac{duP}{dz} = \frac{duR}{dx},\ \frac{duQ}{dz} = \frac{duR}{dy}.$$

Performing the differentiations, these equations become,

$$(438)\quad \left\{\begin{array}{l} u\left(\dfrac{dP}{dy}-\dfrac{dQ}{dx}\right)+\dfrac{Pdu}{dz}-R\dfrac{du}{dx}=o. \\ u\left(\dfrac{dP}{dz}-\dfrac{dR}{dx}\right)+\dfrac{Pdu}{dz}-R\dfrac{du}{dx}=o. \\ u\left(\dfrac{dQ}{dz}-\dfrac{dR}{dy}\right)+Q\dfrac{du}{dz}-R\dfrac{du}{dy}=o. \end{array}\right\}$$

Multiply the first of these by R, the second by — Q, and the third by P, and adding the products we have, after dividing by u,

$$(439)\quad R\frac{dP}{dy}-R\frac{dQ}{dx}+Q\frac{dR}{dx}-Q\frac{dP}{dz}+P\frac{dQ}{dz}-P\frac{dR}{dy}=o.$$

This is an equation of condition, without which (431) cannot be made an exact differential for any value of u.

Suppose we have,

$$(b)\quad (ay-bz)dx+(cz-ax)dy+(bx-cy)dz=o.$$

Here $P=ay-bz$, $Q=cz-ax$, $R=bx-cy$, and (439) is satisfied, but (435) not. Hence equation (b) may be made an exact differential, by multiplying by some factor.

Case 1.

Integrate (431) when condition (439) exists.

As (436) came from differentiating a function of x,y,z, first for one of the variables, as z constant, and then for each of the others in succession as constant, we may, in returning to the integral, consider z as constant, which reduces (436) to

$$(440)\quad Pudx+Qudy=o.$$

The factor u rendering this an exact differential, it is integrable. Let us denote its integral by V, and since the constant added may contain z, let us denote it by z', and the complete integral of (440) may be represented by

$$(441)\quad V+z'=o.$$

Differentiate this for z, as the only variable, and we have, [since V is supposed to contain z],

$$(442)\quad \frac{dVdz}{dz}+\frac{dz'dz}{dz}=o,\quad \text{or,}\ \left(\frac{dV}{dz}+\frac{dz'}{dz}\right)dz=o.$$

As the coefficient of dz in this ought to be the same as the coefficient of dz in (436), we have, by equating these coefficients,

(443) $$Ru = \frac{dV}{dz} + \frac{dz'}{dz}, \quad \text{or } z' = \int \left(Ru - \frac{dV}{dz} \right) dz.$$

This value of z', put into (441), gives for the complete integral of (431), the form,

(444) $$V + \int \left(Ru - \frac{dV}{dz} \right) dz = o.$$

As z' is, by supposition, a function of z, the value of z' in (443) can contain neither x nor y. If the conditions (435) are also fulfilled, then $u = 1$ in (440), 444), &c.

Ex. 1. Integrate the equation,

(c) $$-z^2ydx + z^2xdy + (2zxy + 2x^2)dz = o.$$

This fulfils the condition (439), but not (434). Supposing z constant, (c) becomes,

(d) $$z^2 (xdy - ydx) = o.$$

The part in the vinculum is an exact differential when the factor u is of the value,

(e) $$u = \frac{1}{x^2}.$$

Multiplying by this factor, and integrating, we have for V,

(f) $$V = \frac{z^2y}{x}.$$

To determine z', observe that $R = 2zxy + 2x^2$. Multiply this by (e), and we have,

(g) $$Ru = \frac{2 (zy + x)}{x}.$$

Differentiate (f) for z variable, and we have,

$$\frac{dV}{dz} = \frac{2zy}{x}.$$

These values put into (443), we have,

(h) $$z' = \int 2dz = 2z + C.$$

Hence the complete integral of (c) is,

(k) $$\frac{z^2y}{x} + 2z + C = o.$$

The factor u required to render $\mathrm{P}dx + \mathrm{Q}dy = o$, a complete integral, may be found by the process pointed out in Proposition XIX.

Ex. 2. Integrate the equation,

$$(l) \qquad 2zxdx + 2ydy + x^2dz + adz = o.$$

$$(m) \qquad Ans. \quad x^2z + y^2 + az + \mathrm{C} = o.$$

Case 2.

Integrate (431) when condition (439) does not exist.

When (439) does not exist, we infer that no factor will render (431) an exact differential, and consequently there exists no single primitive whose differential is (431). On this account, when condition (439) does not exist, (431) has been regarded as without meaning; and indeed it is so, if an equation of three variables be regarded as the equation of a surface. But Monge, and after him, writers on the Calculus, consider that when (439) does not exist, (431) may be satisfied by the two equations (441) and (443), in which z' may be taken for any arbitrary function of z. These two equations, taken together, represent a curve of double curvature. Hence in this case, since z' is entirely arbitrary, there are an indefinite number of curves of double curvature which satisfy equation (431).

As an example illustrative of this, take the equation

$$(p) \qquad 2zdx + 2ydy - dz = o.$$

This equation does not satisfy condition (439). Hence, to obtain an integral, take the part of (p) corresponding to (440), viz.

$$(q) \qquad 2zdx + 2ydy.$$

This being an exact integral, (441) becomes

$$(r) \qquad 2zx + y^2 + z' = o,$$

and (443) becomes, (since $\mathrm{R} = -1$),

$$(s) \qquad -1 = 2x + \frac{dz'}{dz}.$$

If z' be any given function of z, as,

$$(t) \qquad z' = mz^2 \text{ then } \frac{dz'}{dz} = 2mz,$$

and the two equations (r) and (s) are,

(2m) $$\begin{cases} 2zx + y^2 + mz^2 = o, \\ 2x + 2mz + 1 = o, \end{cases}$$

which represent a curve of a double curvature.

For any other value of z' we would have another curve, &c.

This is the method of treating (431) when (439) does not exist.

PROPOSITION XXI.

Determine the integral of a partial differential equation of the first order containing three variables.

A partial differential equation of the first order is one derived from an equation between three variables, and may contain the three variables, constants, and the two partial differential coefficients,

$$\frac{dz}{dx}, \text{ and } \frac{dz}{dy},$$

which we will frequently denote by p and q.

We will examine several cases of this proposition.

Case 1.

Let the equation contain but one partial differential coefficient.

If this coefficient equal a constant, we have,

(445) $$\frac{dz}{dx} = m.$$

The integral of this is,

(446) $$z = mx + \varphi y,$$

where ϕy is added because (445) is supposed to be deduced from the primitive, on the supposition that y is constant.

If the partial differential coefficient equal a function of x, we have,

(447) $$\frac{dz}{dx} = \varphi x \therefore z = \int \varphi\, x.dx + \varphi y.$$

If, for example, $\varphi x = ax^2 + bx$, (447) becomes,

$$z = \frac{ax^3}{3} + \frac{bx^2}{2} + \phi y.$$

If the partial differential coefficient equal a function of y, we have,

(448) $$\frac{dz}{dx} = \Psi y,$$

and since this is to be integrated as if y were constant, the integral of (448) is

(449) $$z = x.\Psi y + \phi y.$$

If the partial differential coefficient equal a function of x and y, we have,

(450) $$\frac{dz}{dx} = \Psi(x,y) \therefore z = \int \Psi(x,y)dx + \phi y.$$

The integral of (450) is to be taken for y constant.

Ex.—Integrate the equation,

(a) $$\frac{dz}{dx} = \frac{2ax}{y^2 + x^2} \therefore z = a \log.(y^2 + x^2) + \phi y.$$

Ex. 2. Integrate the equation,

(b) $$\frac{dz}{dx} = \frac{ax}{(y^2 + x^2)^{\frac{1}{2}}} \therefore z = a(y^2 + x^2)^{\frac{1}{2}} + \phi y.$$

If the variables x and z be not separated, we separate them by any of the methods applicable to the particular example.

Illustrative of this, take the following example.

Ex. 3. Integrate the equation,

(c) $$\frac{dz}{dx} = \frac{y^2 - az}{mx}.$$

Separate the variables x and z in this, and we have,

$$\frac{dx}{x} = \frac{mdz}{y^2 - az}.$$

The integral of this is,

(d) $$\log.x = -\frac{m}{a} \log.(y^2 - az) + \phi y.$$

Case 2.

Let the equation contain two partial differential coefficients, and functions of x and y.

We may represent such an equation by

(451) $$\mathrm{P}p + \mathrm{Q}q = o,$$

where P and Q are functions of x and y.

Now since z is to be a function of x and y, we have the general form in Differential Calculus, (377), viz :

(452) $$dz = pdx + qdy.$$

Eliminate p between the two equations (451) and (452), and we have

(453) $$dz = \frac{q}{\mathrm{P}}\,(\mathrm{P}dy - \mathrm{Q}dx).$$

Suppose u the factor which renders $\mathrm{P}dy - \mathrm{Q}dx$ an exact integral, and put

(454) $$u\,(\mathrm{P}dy - \mathrm{Q}dx) = d\mathrm{V}.$$

This put into (453), we have,

(455) $$dz = \frac{qd\mathrm{V}}{\mathrm{P}u}.$$

In order that this may be integrable, we must have $\frac{q}{\mathrm{P}u}$, a function of V, which we may represent by FV; then (455) becomes,

(456) $$dz = \mathrm{FV}d\mathrm{V}.$$

The second side of this being a function of a single variable, is integrable, and we have,

(457) $$z = \int \mathrm{FV}d\mathrm{V} = \varphi\mathrm{V}.$$

We will illustrate this by some examples.

Ex. 1. Integrate the equation,

(a) $$px + qy = o.$$

Here $\mathrm{P} = x$, $\mathrm{Q} = y$, and (454) becomes,

(b) $$d\mathrm{V} = u\,(xdy - ydx).$$

The factor u must obviously be $u = x^{-2}$. Substituting this value into (b), and integrating, we have,

(c) $$\mathrm{V} = \frac{y}{x}.$$

This value of V put into (457), we have,

(d) $$z = \varphi\left(\frac{y}{x}\right),$$

for the complete integral of equation (a).

Ex. 2. Integrate the equation,

(e) $$py - qx = o.$$

Here $\mathrm{P} = y$, $\mathrm{Q} = -x$, and (454) becomes,

(f) $$d\mathrm{V} = u\,(xdx + ydy).$$

The factor u must here obviously be $u = 2$. Put for u this value, and the integral of (f) is,

(g) $$V = x^2 + y^2.$$

This value put into (457), we have,

(h) $$z = \phi(x^2 + y^2),$$

for the complete integral of (e).

In this case, the integration depends upon the integrability of $Pdy - Qdx$; and as this contains but two variables, x and y, this case depends upon Prop. XIII. : for we may put $Pdy - Qdx = o$, and we have a differential equation of two variables.

Case 3:

Let the partial differential equation be of the form,

(458) $$Pp + Qq = R,$$

in which P, Q, and R are functions of x, y, and z. Writing as before, the general form of a differential equation, viz :

(459) $$dz = pdx + qdy,$$

we eliminate p between the two equations (458) and (459), and we have,

(460) $$Pdz - Rdx = q\,(Pdy - Qdx).$$

Now there are two cases of (460), which may be noticed. First, the expressions $Pdz - Rdx$, and $Pdy - Qdx$, may contain only the variables x and y ; or, Second, one or both of these expressions may contain the three variables x,y,z. We will examine each of these cases.

In the first case, we may find a factor u which renders $Pdz - Rdx$ integrable, and a factor u' which renders $Pdy - Qdx$ integrable. Hence assuming

(461) $$u\,(Pdz - Rdx) = dV, \quad \text{and} \quad u'\,(Pdy - Qdx) = dV',$$

(460) becomes,

(462) $$dV = \frac{qu}{u'}\,dV'.$$

This is not integrable unless $\frac{qu}{u'}$ be a function of V′. But as (459) is an exact differential, (462) must also be exact. Hence we may assume $\frac{qu}{u'} = FV'$, and (462) becomes,

(463) $$dV = FV'dV'.$$

The integral of this is,

(463a) $$V = \varphi V',$$

which is the complete primitive of (458).

We will illustrate this by an example.

Ex.—Integrate the equation,

(a) $$px + qy = nz.$$

Here $P = x$, $Q = y$, $R = nz$, and equations (461) become,

(b) $$dV = u\,(xdz - nzdx), \qquad dV' = u'\,(xdy - ydx).$$

The factors which render these integrable are,

$$u = \frac{1}{x^{n+1}} \qquad \text{and} \qquad u' = \frac{1}{x^2}.$$

Multiplying equations (b) by these, and integrating, we have the equations,

$$V = \frac{z}{x^n}, \qquad V' = \frac{y}{x}.$$

These values of V and V′ put into (463), we have for the complete primitive of equation (a),

(d) $$z = x^n.\phi\left(\frac{y}{x}\right),$$

which shows z to be a homogeneous function of the other two variables.

In the second case when x, y and z enter into one or both sides of (460), we may still arrive at the integral by the following considerations, which embrace also the case just considered.

In (459) p and q may be regarded as entirely arbitrary or indeterminate coefficients: for that equation merely indicates that z is a function of x and y; and is the general form of the differential equation of three variables, whatever function z may be of x and y. Hence, q in (460) may be determined by the principle of the method of indeterminate coefficients. Hence, equating the coefficients of the like powers of q, in (460), we have the equations,

(464) $$Pdy - Qdx = o,$$

(465) $$Pdz - Rdx = o.$$

If we were to eliminate q between (458) and (457), we would have, instead of (460), the equation,

(466) $$Qdz - Rdy = p(Qdx - Pdy).$$

For the same reason as before, we may put in this,

(467) $$Qdz - Rdy = o,$$

and the factor of p in (466) gives (464).

Hence, if we can integrate any two of the three equations (464), (465), (467), we can obtain the integral of (458).

For these three equations of condition existing together, if we can integrate one of them as (467), then calling its integral,

(468) $$F(z,y) = C,$$

we have, by solving it, the equation $y = f(z,C)$, and this value of y may be substituted into (465). Equation (465) will then contain only x,z, and the constant C, and may be integrated.

Let its integral be represented by

(469) $$\Psi(x,y,C) + \varphi C = o,$$

where φC is the constant added to complete the integral of (465). This constant ought obviously to be a function of the constant in (468): for the two equations (467) and (465), existing together, the parameter, or arbitrary constant in one of them must be a function of the arbitrary constant in the other. Place the value of C from (468), into (469), and we have an equation containing x, y and z, the primitive of (458).

Ex. 1. Integrate the equation,

(*a*) $$px + qy = m(x^2 + y^2)^{\frac{1}{2}}.$$

Here $P = x$, $Q = y$, $R = m(x^2 + y^2)^{\frac{1}{2}}$, and the three equations of condition (464), (465,) (467), become respectively,

(*b*) $$xdy - ydx = o,$$

(*c*) $$xdz - m(x^2 + y^2)^{\frac{1}{2}}\,dx = o,$$

(*d*) $$ydz - m(x^2 + y^2)^{\frac{1}{2}}\,dy = o.$$

Equation (*b*) is the only one of these immediately integrable. Multiplying it by the factor $u = x^{-2}$ and integrating, we have,

(*e*) $$\frac{y}{x} = C \therefore y = Cx.$$

This value of y put into equation (*c*), we have,

(*f*) $$dz - m(1 + C^2)^{\frac{1}{2}}\,dx = o \therefore z - m(1 + C^2)^{\frac{1}{2}}\,x + \varphi C = o.$$

Put in this the value of the constant C in (*e*), and we have,

$$(g) \qquad z - m(x^2 + y^2)^{\frac{1}{2}} + \phi\left(\frac{y}{x}\right).$$

This is the complete primitive of equation (a).

If two of the equations of condition are integrable, this process is very simple. As another illustration, take the following example.

Ex. 2. Integrate the equation,

$$(h) \qquad pzx - qzy = y^2.$$

Here $P = zx$, $Q = -zy$, $R = y^2$, and the equations (464), (465), and(467), are

$$(k) \qquad zxdy + zydx = o.$$

$$(l) \qquad zxdz + y^2dx = o.$$

$$(m) \qquad zydz + y^2dy = o.$$

Divide (k) by z, and (m) by y, and integrating, we have from (k) the equation.

(n) $\qquad xy = C$, and from (m) the equation,

$$(o) \qquad z^2 + y^2 = \phi.C.$$

Put into (o) the value of C in (n), and we have,

$$(p) \qquad z^2 + y^2 = \phi(xy)$$

for the complete primitive of (h).

Ex. 3. Integrate the equation,

$$-mzp + zxq = xy.$$

The integral is, $y^2 - z^2 = \phi(2ay + x^2)$.

If the three equations of conditions (464), (465), (467), contain each the three variables x, y, and z, we can only obtain the integral by differentiating the two equations (464), (465), and then one of the variables, as x and its two differentials, may be eliminated between the four equations (464), (465), and their differential equations. This will give an equation of the second order, between z and y. If this equation be twice integrated, we will have a relation between z and y, containing two arbitrary constants. By means of this relation, one of the other equations of condition may be integrated, and the relation between x, y, and z, be obtained as before.

PROPOSITION XXII.

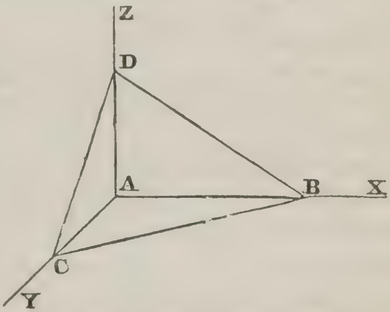
Determine the arbitrary function which enters the integral of a partial differential equation of the first order.

As the determination of the arbitrary constant added in integration, depends upon the nature of the particular problem which gave rise to the differential equation, so the determination of the arbitrary function which enters the integral of a partial differential equation, depends upon the nature of the particular problem which produced the equation.

If this problem be restricted within definite limits, the arbitrary function which enters the integral may be determined by means of these limits. Suppose for example, we have the problem,

Determine the surface whose section with ZY, *is a given curve, and the trace of whose tangent plane on* ZX *makes with the axis of* X *a given angle.*

FIG. 94.



Put m for the tangent of the given angle DBX, and by the second condition of the problem, we have the differential equation,

(470) $$\frac{dz}{dx} = m,$$

the integral of which is,

(471) $z = mx + \varphi y.$

Now to get the section of this surface with the plane ZY, make x zero, and we have, from (471),

(472) $$z = \varphi y.$$

And since the section is, by the first condition of the problem, a given curve, if its equation be,

(473) $$z = fy,$$

then (472) and (473) being the same curve, we have,

(474) $$\varphi y = fy,$$

and (471) becomes,

(475) $$z = mx + fy,$$

where fy is known.

If, for example, the section on ZY were to be a circle, then (473) becomes,

$$z = (R^2 - y^2)^{\frac{1}{2}},$$

and the surface (475) is,

$$z = mx + (R^2 - y^2)^{\frac{1}{2}}.$$

We will give several Geometrical Applications, wherein the process of determining the arbitrary function will be elucidated.

Definition.—Let us, for brevity, call the tangents of the angles DBX and DCY, [fig. 94], which the traces on ZX and ZY of the tangent plane to a surface make with the axis of X and Y, the tangents of X and Y respectively.

APPLICATION 38th.

Determine the surface whose section with the plane ZY is a given curve, and in which the tangent of X varies as a given function of the co-ordinates x and y of the point of tangency.

Recollecting that the partial differential coefficients p and q express the tangents of X and Y respectively, we have, by the second condition of the Proposition, the equation,

$$\frac{dz}{dx} = \phi(x,y). \tag{476}$$

Integrating this, we have,

$$z = \int \phi(x,y)dx + \varphi y. \tag{477}$$

If $\varphi(x,y) = mxy$, this becomes,

$$z = my\frac{x^2}{2} + \phi y. \tag{478}$$

If now the section of this surface, by the plane ZY, be a parabola whose equation is,

$$z = py^2, \tag{479}$$

then (478) becomes,

$$z = my\frac{x^2}{2} + py^2,$$

which is the surface required.

APPLICATION 39th.

Determine the surface whose section with ZY is a given curve, and in which the tangents of X and Y are to each other in a given ratio.

Let m be the ratio of the tangents, then by the Problem, we have,

(481) $$\frac{p}{q} = m.$$

This is the differential equation. The integral of this is, by Proposition XXI. Case 2,

(482) $$z = \varphi(mx + y), \text{ or reciprocally,}$$

(483) $$mx + y = \pi z.$$

Let now the given section on zy be a parabola whose equation is,

(484) $$y = az^2.$$

Since (483) gives for $x = o$, $y = \pi z$, we have $\pi z = az^2$, and (483) gives for the required surface,

(485) $$mx + y = az^2.$$

In this surface all sections parallel to the plane XY are straight lines, and all sections parallel to ZX or ZY are parabolas.

APPLICATION 40th.

Determine the surface whose section with ZY is a given curve, and in which the tangents of X and Y have the same ratio as the co-ordinates x and y of the point of tangency.

By the Problem, we have,

(486) $$\frac{p}{q} = \frac{x}{y} \therefore \frac{dz}{dx} y = \frac{dz}{dy} x.$$

The integral of this is by Proposition XXI,

(487) $$z = \varphi(x^2 + y^2), \text{ or reciprocally,}$$

(488) $$x^2 + y^2 = \pi z.$$

Suppose the section made by the surface on ZY be a parabola whose equation is,

(489) $$y^2 = pz.$$

This is the same section that would be given by making $x = o$, in (488). For $x = o$, (488) becomes $y^2 = \pi z$. Equate this with the value of y^2 in (489), and we have, $\pi z = pz$.

Hence, (488) is,

$$x^2 + y^2 = pz. \tag{490}$$

This is the surface required, and is the paraboloid of revolution; and, indeed, (488) is the general equation of surfaces of revolution, whose axis of revolution is the axis of Z, as is shown in books on Analytical Geometry.

These applications exhibit the mode of determining the arbitrary function that enters the integral of a partial differential equation. If the problem that gave rise to the differential equation be not limited by a sufficient number of conditions, the integrated equation will contain the arbitrary function, without any limit for determining it; but will still express some general truth in respect of all the cases that can be grouped under the given conditions which led to the differential equation. Thus, in Application 40th, if the condition of making a given section with the plane ZY were not given, we could obtain (488), but nothing farther. Now, if z be constant, (488) is a circle. Hence, a property common to all the surfaces that can be grouped under (488), is that all sections perpendicular to the axis of z are circles. For any particular surface of this description, as (490), we must have another condition, [as the first condition in Application 40th,] in order to give a definite form to the arbitrary function in (288).

APPLICATION 41st.

Determine the surface whose section on the plane XY is a given curve and whose tangent plane passes always through a given point.

Take the given point on the axis of z at the distance m from the origin. The equation of the tangent plane is,

$$z' - z = p(x' - x) + q(y' - y), \tag{491}$$

where x,y and z are the co-ordinates of the point of tangency.

For the distance m we have from (491),

$$z - px - qy = m, \quad \text{or,} \quad px + qy = z - m. \tag{492}$$

This is the differential equation. The integral of this is by Case 3, Proposition XXI,

(493) $$\frac{z - m}{x} = \phi\left(\frac{y}{x}\right).$$

To determine the arbitrary function, we have recourse to the other condition of the proposition. Let the section on the plane XY be the hyperbola whose equation is,

(494) $$xy = a^2.$$

This is the same curve that is given from (493), for $z = o$. For $z = o$ (493) becomes,

(495) $$-\frac{m}{x} = \phi\left(\frac{y}{x}\right).$$

Since (494) and (495) are the same curve, let us determine the unknown function in (495). For this purpose, put the quantity involved in ϕ equal an assumed value u, that is, assume,

(496) $$\frac{y}{x} = u.$$

Find the values of x and y from (494) and (496), and we have,

(497) $$y = a\sqrt{u}, \qquad \text{and} \qquad x = \frac{a}{\sqrt{u}}.$$

These values put into (495), we have,

$$\phi u = \frac{-m\sqrt{u}}{a},$$

or restoring the value of u from (496),

(498) $$\phi\left(\frac{y}{x}\right) = -\frac{m}{a}\sqrt{\frac{y}{x}}.$$

This value of the unknown function put into (493), we have for the surface required,

(499) $$a^2(z - m)^2 = m^2xy.$$

This is the hyperbolic cone, whose equation was otherwise determined in Differential Calculus, Proposition LXI.

Without the condition that the surface have a given section on the plane of XY, we could deduce (493), which would appertain to all conic surfaces, whose vertex is on the axis of X, at the distance m from the origin, and which shows that in all such surfaces, the distance $z - m$ equals a homogeneous function of x and y.

If the conic surface (493) had a curve of double curvature for its

directrix, the same plan would determine the arbitrary function. For let the surfaces whose intersection forms the curve of double curvature be

(500) $\quad F(x,y,z) = o, \qquad \text{and} \qquad f(x,y,z) = o.$

Then assuming the terms involved in ϕ equal to u, as in (496), we eliminate x, y, and z from the four equations (493), (496) and (500), and we get,

(501) $$\varphi u = Fu,$$

where Fu is some known combination of u. Restoring in (501) the value of u given in (496), we have a known function of the coordinates x and y, to put into (493).

APPLICATION 42d.

Determine the surface whose section on the plane of XY is a given curve, and whose tangent plane cuts from the axis of Z a line whose length is in a given ratio to the abscissa y of the point of tangency.

Taking (491) for the tangent plane, and putting m for the given ratio, we have by the second condition of the problem,

(502) $$z - px - qy = my.$$

This integrated by Case 3, Proposition XXI., we have,

(503) $$\frac{z}{my} + \log.y = \phi\left(\frac{y}{x}\right).$$

If the section on the plane of XY be a parabola whose equation is, $y = px^2$, (503) becomes,

(504) $$z = my \log.\frac{y}{px^2},$$

the surface required.

PROPOSITION XXIII.

Determine the integral of a partial differential equation of the second order.

Let us, as in the Differential Calculus, assume the partial differential coefficients,

(504a) $\frac{dz}{dx} = p,\ \frac{dz}{dy} = q,\ \frac{d^2z}{dx^2} = r,\ \frac{d^2z}{dxdy} = s,\ \frac{d^2z}{dy^2} = t.$

The most general form of a partial differential equation of the second order is,

(505) $$\varphi(x,y,z,p,q,r,s,t) = o.$$

We will integrate this equation in a few particular cases.

Case 1.

Let (505) contain but one of the partial differential coefficients, r, s, or t, and constants.

We then have, putting m for the constant,

(506) $$\frac{d^2z}{dx^2} = m.$$

This integrated in respect of x, is,

(507) $$\frac{dz}{dx} = mx + \varphi y,$$

where φy is the constant added.

Integrating (507) again, y being constant, we have,

(508) $$z = \frac{mx^2}{2} + x\varphi y + \psi y,$$

where ψy is the constant added at the second integration. Equation (508) is the complete integral of (506). In like manner, if instead of (506), we had the equation,

(511) $$\frac{d^2z}{dx\,dy} = m,$$

as this came from differentiating once for y variable, and once for x variable, we integrate once for y variable, by which we have,

(512) $$\frac{dz}{dx} = my + \varphi x;$$

and this we integrate again for x variable, by which we have,

(513) $$z = mxy + \int \varphi x\, dx + \Psi y.$$

Where Ψy is added to complete the integral.

Case 2.

Let (505) contain but one partial differential coefficient, and x and y.

Solve the equation (505) in this case for the differential coefficent, and we may represent the result by

(514) $$\frac{d^2z}{dx^2} = \text{P},$$

where P is put for a function of x and y.

Integrate this twice for x as the variable, and we have,

(515) $$z = \int\left(\int \text{P}\,dx + \varphi y\right)dx + \Psi y.$$

If the equation to integrate be,

(516) $$\frac{d^2z}{dx\,dy} = \text{P}.$$

This is to be integrated first for one of the co-ordinates, as x variable, and then for the other. Integrating for x variable, we have from (516),

(517) $$\frac{dz}{dy} = \int \text{P}dx + \phi y.$$

Integrating this for y variable, we have,

(518) $$z = \int (\text{P}dx + \varphi y)dy + \phi x.$$

Case 3.

Let (505) be of the form

(519) $$\frac{d^2z}{dx^2} + \text{P}\,\frac{dz}{dx} = \text{Q},$$

where P and Q are each functions of x and y.

If we assume,

(520) $$\frac{dz}{dx} = u.$$

Then differentiating, we have,

$$\frac{d^2z}{dx^2} = \frac{du}{dx},$$

and (519) becomes,

(521) $$\frac{du}{dx} + \text{P}u = \text{Q}.$$

This may be regarded as a partial differential equation in u and

x, and may be integrated for these as the only variables, a function of y being added to complete the integral.

But since P and Q do not contain u, (521) is the same form as (205), whose integral is given at (212). Comparing (521) and (205), we may write the integral of (521), taking care to add in the integral (212), φy for C. This integral in conjunction with (520) makes known the relation between x,y and z, involved in (519).

Case 4.

Let (505) be of the form,

(522) $$\frac{d^2z}{dx\,dy} + \mathrm{P}\frac{dz}{dx} = \mathrm{Q}.$$

Differentiate the assumed equation (520) for y, and by substitution (522) becomes,

(523) $$\frac{du}{dy} + \mathrm{P}u = \mathrm{Q}.$$

The same in form as (521), and integrated as the linear equation (205).

Case 5.

Let (505) be in its general form.

We may by examining it in this form, deduce a process for integrating it, in many particular cases. Suppose we have the equation

(524) $$\mathrm{R}r + \mathrm{S}s + \mathrm{T}t = \mathrm{U},$$

where R, S, T and U contain x,y,z,p, and q, in any manner. The general form of the first differential equation is,

(526) $$dz = pdx + qdy,$$

and as in Differential Calculus, at (399), we have,

(527) $$dp = rdx + sdy, \quad \text{and} \quad dq = sdx + tdy.$$

We may eliminate t and r between equations (527) and (524), and we have,

(528) $$\mathrm{R}dpdy + \mathrm{T}dqdx - \mathrm{U}dxdy = s(\mathrm{R}dy^2 - \mathrm{S}dxdy + \mathrm{T}dx^2).$$

Since the coefficients r, s, and t in (527), are indeterminate, we may equate the co-efficients of the like powers of s in (528), which leads to the conditions,

(529) $$\mathrm{R}dpdy + \mathrm{T}dqdx - \mathrm{U}dxdy = 0.$$

(530) $$R dy^2 - S dx dy + T dx^2 = o.$$

Equations (529), (530), and (528) are analogous to (460), (464) and (465), and depend upon the same principle.

As (530) contains dx and dy in the second power, we may solve it for $\frac{dy}{dx}$, and putting m and m' for the roots, we have the two equations,

(531) $$dy - m dx = o, \quad \text{and} \quad dy - m' dx = o.$$

These two values of dy substituted successively into (529), we have the two sets of equations,

(532) $$dy - m dx = o, \qquad R m dp + T dq - U m dx = o.$$

(533) $$dy - m' dx = o, \qquad R m' dp + T dq - U m' dx = o.$$

These equations exist in conjunction with (526).

If we can integrate the equations (532), call their integrals V and V', then as at (460), (463), we have,

(534) $$V = \varphi V'.$$

This is the first integral of (524).

In like manner, if we could integrate equations (533), we would, by calling their integrals L and L', have for the first integral of (524),

(535) $$L = \varphi' L'.$$

If now we integrate either of the equations (534) or (535), we will have the original function of (524). As the equations (534), (532) exist together, we may, if convenient, employ either of equations (532) to aid in the integration of (534).

If we can eliminate p and q between (534), (535), and (526), the result is an exact differential, because (526) is an exact differential, and the integral of this resulting equation may be obtained as in Proposition XVIII.

As an example, take the equation,

(b) $$r - a^2 t = o.$$

Eliminating r and t from this by means of (527), and equating the like powers of s, we get as particular forms of (529), (530), the equations,

(c) $$dy^2 - a^2 dx^2 = o, \quad \text{and} \quad dy dp - a^2 dq dx = o.$$

The equations (532), (533), become in this case,

(d) $dy - adx = o, \qquad dp - adq = o,$ and

(e) $dy + adx = o, \qquad dp + adq = o.$

The integrals of equations (d) are, since a is constant,

(f) $y - ax = V', \qquad$ and $\qquad p - aq = V.$

Hence (534) is,

(g) $$p - aq = \varphi(y - ax).$$

The integrals of equations (e) are,

(h) $y + ax = L' \qquad$ and $\qquad p + aq = L,$

and (535) becomes,

(k) $$p + aq = \varphi'(y + ax).$$

As we can here eliminate p and q between equations (526), (g), and (k), we have, after performing this elimination, the result,

(l) $$dz = \frac{\varphi'(y + ax)\,(dy + adx) + \varphi(y - ax)\,(dy - adx)}{2a}.$$

This equation falls under the process of Proposition XVIII. We may write its general form by observing, that since the integral of the form $\varphi u du$, is some function of u, which may be represented by fu, so, putting u for $y + ax$, or $y - ax$, in (l), we may write the integral of (l) in the form,

(m) $$z = f(y + ax) + f'(y - ax),$$

where the denominator $2a$ of (l), and the constant C, are included in the functions f and f'.

Equation (b) is the differential equation of vibrating cords, and occasioned, at one time, much discussion.

As another example, integrate the equation,

(n) $$x^2r + 2xys + y^2t = o.$$

Proceeding as before, we find the values of m and m' to be equal, and equations (532) become,

(o) $$xdy - ydx = o, \quad xdp + ydq = o.$$

The integrals of these are,

(2b) $$\frac{y}{x} = V', \quad p + V'q = V,$$

consequently (534) becomes,

(2c) $$p + q\frac{y}{x} = \varphi\left(\frac{y}{x}\right).$$

This is the first integral, and $(2c)$ is to be integrated as equation (458).

Comparing equation $(2c)$ with (458), the equations (464) and (465), become, in this case,

$$(2d) \qquad xdy - ydx = o, \quad dz - \varphi\left(\frac{y}{x}\right)dx = o.$$

The integral of the first of these is $y = Cx$. This value of y put into the second of $(2d)$, the integral of that equation is,

$$(2e) \qquad z - x\phi C = \Psi C,$$

or restoring in this the value of C from $y = Cx$, we have for the complete integral of (n),

$$(2f) \qquad z = x\varphi\left(\frac{y}{x}\right) + \Psi\left(\frac{y}{x}\right).$$

As another example, integrate the equation,

$$(2g) \qquad q^2r - 2pqs + p^2t = o.$$

Proceeding with this as in Example (n), we find the values of m and m' to be equal, and the integration is performed as in Ex. (n).

The result may be put in the form

$$(2h) \qquad y = x\phi z + \psi z.$$

As two arbitrary functions enter the integral of a partial differential equation of the second order, we must, in order to determine these two functions subject the problem that produced the differential equation to two conditions besides the condition that produced the differential equation. For example, if we had the proposition,

Determine the surface whose sections on the planes of ZY *and* XY *are given curves, and whose second partial differential coefficient at any point is constant*, we could by the last condition form the differential equation of the Problem, viz.

$$(536) \qquad \frac{d^2z}{dx^2} = m.$$

The integral of this is,

$$(537) \qquad z = \frac{mx^2}{2} + x\phi y + \psi y.$$

Let the section on XY be a straight line, whose equation is $y = ax$, and the section on ZY be a parabola, $z = by^2$. Then, when $x = o$ (537) becomes $z = \psi y$, and since this is the same curve as $z = by^2$, we have,

(538) $$\psi y = by^2.$$

Again when $z = o$, (537) becomes, $mx^2 + 2x\phi y + 2\psi y = o$, which is the same as the straight line $y = ax$. Eliminate x between these two equations and we have

(539) $$my^2 + 2ay\phi y + 2a^2\psi y = o.$$

The values of ϕy and ψy from the two equations (538), (539) put into (537) we have for the particular surface required

(540) $$2az = amx^2 - xy\,(2a^2b + m) + 2aby^2.$$

This will serve to show the method of determining the arbitrary functions that appear in the integrals of equations of the second order.

PROPOSITION XXIV.

Determine the integral of a partial differential equation of the first order containing four variables.

If we have the equation,

(541) $$z = \phi(x,y,u),$$

we may, by putting p, q and s, for the partial differential coefficients, represent its differential by

(542) $$dz = pdx + qdy + sdu.$$

If now we have the partial differential equation,

(543) $$\mathrm{N}s + \mathrm{P}p + \mathrm{Q}q = \mathrm{R},$$

in which N, P, Q, and R, are functions of x,y,z and u, we may eliminate one of the coefficients as s between (543) and (542), and we have,

(544) $$\mathrm{N}dz - \mathrm{R}du = p(\mathrm{N}dx - \mathrm{P}du) + q(\mathrm{N}dy - \mathrm{Q}du).$$

Since p, q, and s, in (542), are indeterminate coefficients, p and q, may be regarded as indeterminate in (544), and in order to determine them, we may put as in (460), the coefficients of p and q in (544), equal to zero, by which we have the three equations,

(545) $$\mathrm{N}dz - \mathrm{R}du = o, \quad \mathrm{N}dx - \mathrm{P}du = o, \quad \mathrm{N}dy - \mathrm{Q}du = o.$$

If these equations become integrable by being multiplied by the factors m, m', m'' respectively, then by putting $d\mathrm{V}, d\mathrm{V}', d\mathrm{V}''$, for equations (545) respectively, and substituting into (544), we have the equation,

(546) $$dV = \frac{pm\,dV'}{m'} + q\,\frac{m}{m''}\,dV''.$$

This being an exact differential, we must have $p\,\dfrac{m}{m'}$, a function of V', and $q\,\dfrac{m}{m''}$ a function of V''. Consequently the integral of (546) is of the form,

(547) $$V = \varphi V' + \psi V'',$$

or since the two terms on the second side of (546) may be regarded as the two terms of a differential of a function of two variables, V' and V'', we may, instead of (547), write the form,

(548) $$V = \varphi(V', V'').$$

As the equations (545) exist together, we may obviously combine them in any manner with each other, and with the integrals of each other, which will enable us in general to integrate (543), when we can integrate any one of (545).

The equations (545) may thus be treated in the same manner as the equations (464), (465), (467).

Ex.—Integrate the equation,

(*a*) $$us + xp + yq = z.$$

Here $N = u$, $P = x$, $Q = y$, $R = z$, and equations (545) become,

(*b*) $$udx - xdu = o, \quad udz - zdu = o, \quad udy - ydu = o.$$

The integrals of these are,

(*c*) $$\frac{x}{u} = V, \frac{z}{u} = V', \frac{y}{u} = V'',$$

and (547) becomes,

(*d*) $$\frac{x}{u} = \varphi\left(\frac{z}{u}\right) + \psi\left(\frac{y}{u}\right),$$

where any one of the variables is a homogeneous function of the other three.

PROPOSITION XXV.

Determine the integral of a partial differential equation of the first order, containing three variables, and exceeding the first degree.

The general form for the partial differential equation of a function of three variables, is,

(549) $$dz = pdx + qdy.$$

The second side of this being an exact differential, must fulfil the condition of exact differentials at equation (397), viz:

(550) $$\frac{dp}{dy} = \frac{dq}{dx}.$$

But inasmuch as p and q, in (549), may contain z, we must differentiate p in regard to both y and z, and q in regard to both x and z. This changes (550) to

(551) $$\frac{dp}{dy} + q\frac{dp}{dz} = \frac{dq}{dx} + p\frac{dq}{dz}.$$

If now,

(552) $$\mathrm{P} = o,$$

be a partial differential equation of three variables, containing p and q, involved to any power, as p^2, q^2, pq, &c., then since (551) is the equation of condition to which (552) is subject, in order that (552) may be an exact differential, we eliminate p between (552) and (551), and we have a differential equation containing the four variables x,y,z,q, and their partial differential coefficients in respect of q. The result of this elimination may be represented by

(553) $$\varphi\left(x,y,z,q\ \frac{dq}{dz},\ \frac{dq}{dy},\ \frac{dq}{dx}\right) = o.$$

This equation may be integrated by Proposition XXIV., as an equation of four variables, and its integral being solved for q, we may represent it by

(554) $$q = \mathrm{F}\,(x,y,z,\mathrm{C}),$$

where C is the arbitrary constant introduced by integrating (553). From (554) and (552), we can eliminate q, and get the value of p, which we may represent by

(555) $$p = \mathrm{F}'\,(x,y,z,\mathrm{C}).$$

These values of p and q put into (549), we have,

(556) $$dz = \mathrm{F}'\,(x,y,z,\mathrm{C})dx + \mathrm{F}\,(x,y,z,\mathrm{C})dy.$$

This being a total differential equation of three variables, may be integrated by Proposition XX. We may represent its integral by

(557) $$\psi(x,y,z,C) = C'.$$

The constant C′ may be a function of the constant C introduced in the integral of the equation of condition (553); but of this more hereafter.

To reduce this process to a more practical form, suppose we differentiate (552) for x,y,z,p and q, variables, and represent its differential by

(558) $$dP = Adx + Bdy + Cdz + Ddp + Edq = o.$$

Assume,

(559) $$\frac{dq}{dx} = p', \ \frac{dq}{dy} = q', \quad \text{and} \quad \frac{dq}{dz} = s', \quad \text{and}$$

(560) $$dq = p'dx + q'dy + s'dz.$$

Since p and q in (552) are functions of x, y and z, we must, to obtain the partial differential coefficients,

$$\frac{dp}{dy}, \quad \text{and} \quad \frac{dp}{dz},$$

in (551), first put z and x constant in (552), and then x and y constant. This is the same as putting first $dx = o$, and $dz = o$, in (558), by which we have, from (558),

(561) $$\frac{dp}{dy} = -\frac{B + Eq'}{D};$$

and next, putting in (558), $dx = o$, $dy = o$, by which we have,

(562) $$\frac{dp}{dz} = -\frac{C + Es'}{D}.$$

The values (561) and (562), put into (551), we have,

(563) $$Dp' + s'(Dp + Eq) + Eq' = -(B + Cq).$$

Eliminate p' between (563) and (560), and putting the coefficients of s' and q' in the resulting equation equal to zero [according to Proposition XXIV, (545)], we have the equations,

(564) $$\begin{cases} Ddy - Edx = o, \\ Ddz - (Eq + Dp)\,dx = o, \\ Ddq + (B + Cq)\,dx = o. \end{cases}$$

From these equations we can eliminate p by means of (552), and equations (564) may then be integrated as (545) in last Proposition.

The equations (564) exist in conjunction with (549;) and, indeed, if we eliminate E between the two first of (564), we get (549).

It frequently happens that the last of (564) is immediately integrable, in which case the other two are not needed.

We will give a few examples illustrative of this theory.

Ex. 1. Integrate the equation,

$$(a) \qquad xp - aq^2 - z = o.$$

Differentiate this, and comparing the differential with (558), we find $A = p$, $B = o$, $C = -1$, $D = x$, $E = 2aq$, and equations (564) become,

$$(b) \; xdy + 2aqdx = o, \; xdz + (2a + 1)\,qdx = o, \; xdq - qdx = o.$$

The last of these is immediately integrable, and gives us,

$$(c) \qquad q = Cx.$$

By means of equations (a) and (c), eliminate p and q from (549), and we have,

$$(d) \qquad dz = \left(\frac{z}{x} + ab^2x\right) dx + Cxdy,$$

which is to be integrated by Proposition XX. Performing this integration, we will have the required relation between x, y and z.

We will now examine several cases in which equations (564) can be integrated.

If we put the last of equations (564) in the form

$$(565) \qquad dq + \left(\frac{B}{D} + \frac{Cq}{D}\right)dx = o,$$

it is obvious that it will be immediately integrable when the proposed equation is such that the coefficient of dx in (565) contains only q and x. In other words, that (565) may be immediately integrable, we must have,

$$(566) \qquad \frac{B}{D} + \frac{Cq}{D} = \varphi(x,q),$$

where φ denotes any given function.

Whenever (566) exists, the value of q may be obtained immediately by integrating (565.)

Again, if we eliminate dx between the first and last of (564), we have,

$$(567) \qquad dq + \left(\frac{B}{E} + \frac{Cq}{E}\right) dy = o,$$

which is immediately integrable if the proposed equation (552) be such that we have,

(567a) $$\frac{\mathrm{B}}{\mathrm{E}} + \frac{\mathrm{C}q}{\mathrm{E}} = \varphi(q,y).$$

Whenever this equation (567a) exists, the value of q may be obtained immediately by integrating (567).

Again, if we eliminate dx between the two last of (564), we have,

(568) $$dq + \left(\frac{\mathrm{B} + \mathrm{C}q}{\mathrm{D}p + \mathrm{E}q}\right) dz = o,$$

which is immediately integrable if the proposed equation (552) be such that we have,

(569) $$\frac{\mathrm{B} + \mathrm{C}q}{\mathrm{D}p + \mathrm{E}q} = \phi(q,z).$$

Ex. 2. Integrate the equation,

(e) $$p - qy + q^2x + z = o.$$

This leads to the condition,

(f) $$dq = o, \quad \therefore \quad q = \mathrm{C},$$

and (e) becomes immediately integrable.

As another example, take the equation,

(g) $$p - yq^2 + qx - xz = o.$$

This gives for the last of (564),

(h) $$dq - (q^2 + x)dx = o,$$

which is immediately integrable.

Ex. 3. Integrate the equation,

(k) $$ax - q^2y - p^2x + z = o.$$

For this example, (567a) exists, and (567) becomes,

(l) $$dq - (1 - q)\,\frac{dy}{2y} = o,$$

which is immediately integrable.

Besides the cases when one of the equations (566), (567a), and (569) exist, we may frequently integrate two of the equations (564) together, when these two contain between them but three of the four variables, q, x, y, and z. The method of integrating in such a case is pointed out at the end of Proposition XXI., for the equations (464), (465), and leads generally to an equation of the second order.

Another plan is to integrate the three equations (564) together.

Again, if we assume,

(570) $$v = px + qy - z,$$

we have an extensive class of partial differential equations of the form,

(571) $$\phi(x,y,p,q,v) = o.$$

If we differentiate (570), we have, by virtue of the equation,

(572) $$z = px + qy,$$ the result,

(573) $$dv = xdp + ydq.$$

If we regard p and q as the independent variables, x and y as the differential coefficients, and v as the function sought, we may integrate (571) and (573) as we integrated in Proposition XXI.; that is, eliminate one of the co-ordinates, as x, between (571) and (573), and from the resulting equation form two equations of condition, by equating the coefficients of the like powers of y, which may be integrated as we integrated (464), (465). If we call the integrals of these two equations of condition m and n, we may eliminate p and q between the equations m, n, and (571). The result will be the integral required, and will contain two arbitrary constants.

Illustrative of the plan of integrating (571), take the following example.

Ex.—Integrate the equation,

(2a) $$px + q^2y - v = o.$$

Eliminate x between (2a) and (573), and we have,

(2b) $$pdv - vdp = y\,(pdq - q^2dp).$$

Equating the coefficients of the like powers of y in (2b), we have the equations of condition

(2c) $$pdv - vdp = o, \qquad pdq - q^2dp = o,$$

whose integrals are,

(2d) $$\frac{v}{p} = m, \qquad \log.p + \frac{1}{q} = n.$$

Eliminate p, q, and v between (2d), (2a), and (570), the result is the integral of (2a).

If the proposed equation be of the form

(574) $$v = \mathrm{F}\,(p,q),$$

then since this does not enable us to eliminate x or y from (573), we may put the coefficients of x and y in (573) separately zero, by which we have the equations,

(575) $dp = o, \quad dq = o, \quad \therefore \quad p = m, \quad \text{and} \quad q = n,$
and (574) becomes,

(576) $$z - mx - ny = \mathrm{F}\,(m,n).$$

Hence we see that the form (574) is integrated by putting the constants m and n for p and q, in (574).

The relation between the constants m and n, and their determination, we will point out in the next Proposition.

The integration of partial differential equations may be frequently simplified by introducing an indeterminate coefficient. Suppose the equation be,

(577) $$\mathrm{F}\,(p,x) = \mathrm{F}'\,(q,y).$$

By putting $\mathrm{F}\,(p,x) = \omega$, we have also $\mathrm{F}'\,(q,y) = \omega$, and these two equations being solved for p and q, we have $p = f\,(x,\omega)$, and $q = f'\,(y,\varphi)$. These values of p and q put into (549), we have,

(578) $$dz = f\,(x,\omega)dx + f'\,(y,\omega)dy.$$

If ω can be taken as constant, this may be integrated, and we have,

(579) $$z = \int f\,(x,\omega)dx + \int f'\,(y,\omega)dy + \mathrm{C},$$

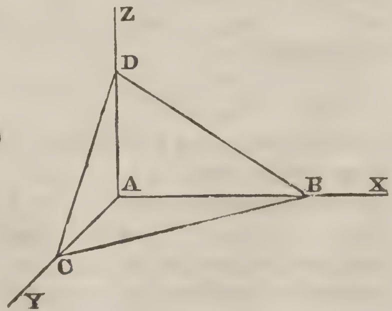
where C may be a function of ω.

We will now give some Geometrical Applications involving differential equations of the higher degrees.

APPLICATION 43d.

Determine the surface whose tangent plane passes through a given point, and cuts off, with the co-ordinate planes, a given pyramid.

FIG. 95.



Let the given point be on the axis of Z.

Let the equation of the tangent plane be
$z' - z = p\,(x' - x) + q(y' - y)$
where x', y', z', are the co-ordinates of any point on the plane.

For the intercepts of this plane with the axes, we have,

(580) $AB = -\frac{z-px-qy}{p}$, $AC = -\frac{z-px-qy}{q}$, $AD = z-px-qy$.

Put b^3 for the solid content of the given pyramid, and a for the distance AD, and by the problem we get the equation

(580a) $$z - px - qy = \left(\frac{6b^3pq}{a}\right)^{\frac{1}{2}}.$$

This being of the form (574), the complete primitive is,

(581) $$z - mx - ny = \left(\frac{6b^3\,m\,n}{a}\right)^{\frac{1}{2}}.$$

We will examine this result further in the next Proposition.

APPLICATION 44th.

Determine the surface whose tangent plane cuts off, with the co-ordinate planes, a given pyramid.

Putting s^3 for six times the volume of the pyramid, and using the intercepts (580), we have, by the problem,

(581) $$z - px - qy = s\,(pq)^{\frac{1}{3}},$$

which being of the form (574), its integral is,

(582) $$z - mx - ny = s\,(m\,n)^{\frac{1}{3}}.$$

We will examine this further in the next Proposition.

APPLICATION 45th.

Determine the surface whose tangent plane cuts off, on the plane of XY, a triangle whose area varies as the square of the sum of the co-ordinates x and y of the point of tangency.

Taking the notation (580), and putting for the area, $e\,(x + y)^2$, we have, by the problem,

(583) $$z - px - qy = (x + y)\,(2epq)^{\frac{1}{2}},$$

where e is any constant. As this is of the form (571), then by eliminating x between (583) and (573), and equating in the resulting equation the coefficients of the like powers of y, we have the equations,

(584) $$-\,dp + dq = o, \quad (2epq)^{\frac{1}{2}}\,dv - vdp = o,$$

whose integrals are,

$$(585) \qquad q - p = m, \quad \log.v - \frac{1}{(2e)^{\frac{1}{2}}}\int \frac{dp}{(p^2 + pm)^{\frac{1}{2}}} = n.$$

Perform the integration of the last of (585), and then eliminate p, q, and v between (585), (583), (570), and we have the primitive of (583), which makes known the surface required.

APPLICATION 46th.

Determine the surface which cuts a given family of surfaces at a given angle.

This is analogous to Application 26th. By a family of surfaces is understood all the surfaces that can be expressed by a single equation, when different values are given to a parameter in that equation. Thus the family of paraboloids of revolution may be expressed by the equation,

$$(586) \qquad x^2 + y^2 = \beta z,$$

which, when β takes different values, gives the family of paraboloids of revolution.

Let the equation of the surfaces cut be represented by

$$(587) \qquad \mathrm{F}\,(x',y',z',\mathrm{B}) = o,$$

and let the surface required be,

$$(588) \qquad f\,(x'',y'',z'') = o.$$

The angle of two intersecting surfaces is the same as the angle formed by their tangent planes at any point of their common section. Let

$$(589) \qquad z - z' = p\,(x - x') + q\,(y - y'),$$

be the tangent plane to (587), and

$$(590) \qquad z - z'' = p'\,(x - x'') + q'\,(y - y''),$$

be the tangent plane to (588).

If m be the angle which these planes make with each other, we have, by Analytical Geometry,

$$(591) \qquad \mathrm{Cos}.m = \frac{1 + pp' + qq'}{(1 + p^2 + q^2)^{\frac{1}{2}}\,(1 + p'^2 + q'^2)^{\frac{1}{2}}}.$$

Supply in this the coefficients p and q from the given surface (587), and then eliminating B between (587) and (591), we have the differential of the surface required.

If the angle m be a right angle, then (591) becomes,

(592) $$1 + pp' + qq' = o,$$

and the integration is effected by Proposition XXI.

As an example of the integration find the surface which cuts at a given angle all the cylinders expressible by the equation,

(a) $$x^2 + y^2 = R^2.$$

Here $p = o$, $q = o$, and (591) becomes,

(b) $$\text{Cos.}m = \frac{1}{(1 + p'^2 + q'^2)^{\frac{1}{2}}}.$$

From this we get,

(c) $$p'^2 + q'^2 = C^2.$$

where we put, for brevity, C^2 for $\dfrac{1 - \cos.^2 m}{\cos.^2 m}$.

The process pointed out at (577) will apply here; for if we put $p' = C \cos.m$ and $q' = C \sin.m$, equation (c) is satisfied. Hence, (578) becomes,

(d) $$dz = C \cos.m\, dx + C \sin.m.dy,$$

which since m may be taken constant, gives for the integral,

(e) $$z = Cx \cos.m + Cy \sin.m + n.$$

The same result could be obtained from (564) by observing, that in this case, the last of (564) gives $dq' = o \therefore q' =$ a constant, which put into equation (c), gives $p' =$ a constant.

The integrations in the following examples, under this Application, are performed by Proposition XXI.

Ex. 2. Find the surface which cuts at right angles all the family of paraboloids whose equation is,

(f) $$x^2 + y^2 = 4az.$$

(g) $$\text{Ans.} \quad 2z^2 + x^2 + y^2 = 2\phi\left(\frac{y}{x}\right).$$

Ex. 3. Find the surface which cuts at a right angle the spheres whose equation is,

(h) $$x^2 + y^2 + z^2 = R^2.$$

(k) $$\text{Ans.} \quad \frac{z}{x} = \phi\left(\frac{y}{x}\right),$$

any conic surface whose vertex is at the centre.

Ex. 4. Find the surface which cuts at a right angle all the spheres which touch a given plane at a given point.

Let the plane XY be the given plane. Take the given point for the origin, and for the family of spheres we have the equation.

$$(l) \qquad x^2 + y^2 + z^2 = 2\mathrm{R}z,$$

and for the trajectory required, we get,

$$(m) \qquad x^2 + y^2 + z^2 = x\varphi\left(\frac{y}{z}\right).$$

APPLICATION 47th.

Given the equation of a surface, determine the equation of an equivalent surface.

Definition.

Equivalent surfaces are those on which if equivalent areas be taken, the projections of these areas on a given plane will be equal.

Let the plane of XY be the plane of projection; and let

$$(593) \qquad z = \varphi(x,y),$$

be the given surface; and suppose that

$$(594) \qquad z = \psi(x,y),$$

be the required equivalent surface.

Putting p' and q' for the differential coefficients of (593), and p and q for the differential coefficients of (594), and equating the areas of the equivalent surfaces as given by (167), we have,

$$(595) \qquad \int\int dxdy(1 + p'^2 + q'^2)^{\frac{1}{2}} = \int\int dxdy(1 + p^2 + q^2)^{\frac{1}{2}}.$$

Since the projections on xy of equivalent areas of the surfaces are equal, the $dxdy$ of (595), is the same for both surfaces. Hence, omitting the signs of integration, and dividing by $dy\ dx$, (595) becomes,

$$(596) \qquad p'^2 + q'^2 = p^2 + q^2.$$

The values of p' and q' can be supplied in terms of x and y, from the given surface (593), and (596) will give for determining the equivalent surface,

$$(597) \qquad p^2 + q^2 = \mathrm{F}(x,y).$$

The integral of this will give the required surface (594).

Ex. 1. Let the given surface (593), be the paraboloid of revolution.

Here (593) becomes,

$$(a) \qquad 4mz = x^2 + y^2,$$

and equation (597) becomes,

$$(b) \qquad p^2 + q^2 = \frac{x^2 + y^2}{4m^2},$$

which, when integrated, will be the equivalent surface of (a). The integration belongs to Proposition XXV.

Ex. 2. Let (593), the given surface, be a plane.

Taking for the equation of the given plane,

$$(c) \qquad z = mx + ny + \mathrm{B},$$

(597) becomes,

$$(d) \qquad p^2 + q^2 = (m^2 + n^2)^{\frac{1}{2}} = \mathrm{C}^2,$$

a constant. This is integrated as equation (c), in Application 46th, and gives a plane.

It is obvious, from the definition of equivalent surfaces, that if a right cylinder be erected on the plane XY, it will cut equal areas from the two surfaces.

PROPOSITION XXVI.

Determine the singular solution of a partial differential equation of the first order.

In Proposition XV., we observed that the singular solution of a differential equation of two variables depended upon the principle for determining the Locus of the intersection of consecutive lines. A similar remark applies to differential equations of three variables. In the preceding proposition we have seen, that in determining the complete primitive of a partial differential equation exceeding the first degree, two constants were introduced, as at (554) to (557), or at (570) to (576). Suppose m and n be these constants. The conditions of the problem that gave rise to the partial differential equation may be such that we can determine one of these constants

in terms of the other, so that, putting $m = \varphi n$, the integrated equation instead of being of the form,

(598) $$F(x,y,z,m,n) = o,$$

will be of the form,

(599) $$F(x,y,z,n,\varphi n) = o.$$

The conditions of the problem may, however, be such that no relation can be established between m and n; in which case they will be independent of each other. If the relation $m = \varphi n$, can be established from the conditions of the problem, the parameter n may enter into (599) in such a manner that we can eliminate it by the principles of consecutive surfaces, as detailed in the Differential Calculus, Proposition LIX.; that is, we may eliminate n between (599) and its differential for n variable. The result of this elimination will be a surface which may be represented by

(600) $$\varphi(x,y,z) = o.$$

This surface, as is shown in the Differential Calculus, Proposition LIX., Cor. 3d, envelopes all the surfaces (599). The result (600) is called the singular solution of (599).

If the parameters m and n are independent of each other, we may differentiate (598) first for m variable, then for n variable, and we have the equations,

(601) $$\frac{dFdm}{dm} = o, \qquad \frac{dFdn}{dn} = o,$$

where, for brevity, F is put for F (x,y,z).

If m and n be eliminated between (598) and (601), the result will be an equation of the form (600), which will be the envelope of all the surfaces comprised in (598).

The result (600) is in this case called the singular solution of (598).

We will illustrate this by a few particular examples.

For this purpose, resume Application 43d, which proposed to *Determine the surface whose tangent plane passes through a given point on the axis of* Z, *and cuts off, with the co-ordinate planes, a given pyramid.*

The conditions of the problem gave for the integral of the differential (580a), the equation,

(602) $$z - mx - ny = \left(\frac{6b^3mn}{a}\right)^{\frac{1}{2}}.$$

Now the condition of passing through a given point on the axis of Z gives us, [since in (581), $p = m$, and $q = n$], the relation,

(603) $$z - mx - ny = a.$$

Hence,

(604) $$a = \left(\frac{6b^3mn}{a}\right)^{\frac{1}{2}}, \quad \therefore \quad m = \frac{a^3}{6b^3n} = \varphi n.$$

This is the φn which we are to substitute into (602) for m. Making this substitution (602) becomes,

(605) $$z - \frac{a^3}{6b^3n}x - ny = a.$$

Differentiating this for n as the only variable, we have,

(606) $$\frac{a^3x}{6b^3n^2} - y = o.$$

Eliminate n between (605) and (606), and we have,

(607) $$3b^3(z - a)^2 = 2a^3\,xy.$$

This is the singular solution of (602). It is obvious that the process here is precisely that given at Proposition LXI, Differential Calculus, where the same problem is solved, and the result (607) obtained.

The surface (607) is a single surface, while (602) the complete primitive is a family of planes which are all touched by (607). If, instead of taking the point through which the tangent plane passes to be on the axis of z, we had taken its co-ordinates $(a',b',c',)$ the solution would be obtained in the same manner. As an illustration of the case where no relation can be fixed between m and n by the conditions of the problem that produced the differential equation, resume Application 44th, which proposed to *Determine the surface whose tangent plane cuts off with the co-ordinate planes a given pyramid.* The conditions of the problem gave a differential equation whose integral (582), is

(608) $$z - mx - ny = s(m\,n)^{\frac{1}{3}}.$$

Here there is no condition of the problem which enables us to fix any relation between m and n. These parameters, therefore, are independent of each other. Hence, differentiating (608) first for m variable, and then for n variable, we have for (601), the equations,

(609) $-\,xdm = \frac{s}{3}\left(mn\right)^{-\frac{2}{3}} ndm, \quad -\,ydn = \frac{s}{3}\left(mn\right)^{-\frac{2}{3}} mdn.$

Eliminate m and n between the three equations, (608) and (609), and we have,

(610) $$xyz = \frac{s^3}{27}.$$

This is the singular solution of (608), and is the surface to which all the surfaces (608), are tangent.

This singular solution is the Proposition LXIII., Differential Calculus, where the same result is obtained.

If the quantities m and n both disappear when the equation is differentiated for them as variables, we conclude such a primitive does not admit of a singular solution.

If the conditions of the problem which produced the complete primitive do not enable us to fix a relation between m and n, it might occur that one of them, as n, would disappear from (601), and the other, m, remain. In this case we can eliminate m from the complete primitive, and the result will be an equation which will satisfy the proposed differential equation.

As an example, resume Application 46th. The first example under that application proposed to *Determine a surface which would intersect at a given angle a family of cylinders.* Equation (c) of that application is,

(611) $$p^2 + q^2 = c^2.$$

This is integrated at (e) in that Application, or we may integrate it in the usual way by equations (564), which leads to $q = m$, a constant; hence, $p = (c^2 - m^2)^{\frac{1}{2}}$.

These values of p and q put into the equation $dz = pdx + qdy$, we have, by integrating, and adding the constant n,

(612) $$z = (c^2 - m^2)^{\frac{1}{2}}\, x + my + n.$$

As the problem furnished no condition for establishing a relation between m and n, we may consider them independent. If we differentiate (612) for m variable, we get,

(613) $$o = \frac{-\,mx}{(c^2 - m^2)^{\frac{1}{2}}} + y.$$

The differential of it for n, leads to no result, as n disappears in the differential equation.

Eliminate m between (612) and (613), and we get,

(614) $$(z - n)^2 = c^2(x^2 + y^2),$$

the equation of a cone with its vertex on Z, and the axis of Z for its axis. The vertical angle of the cone depends on the given quantity c, which is a function of the given angle of intersection of the surfaces. Equation (614) contains the arbitrary constant n which determines the position of the vertex. We would obtain the same result by differentiating, for m variable, the integral (e), of Application 46th, and eliminating m from (e), and its differential.

As (614) contains but one arbitrary constant n, it may be regarded as the general integral of (611). The values of p and q deduced by differentiating (612) satisfy (611); and the values of p and q deduced from (614), and put into (611), also satisfy (611).

The relation between the surfaces (612) and (614), is obvious: for if n be made constant, and m be left arbitrary or variable, (612) is a system of planes passing through a point on the axis of z, each plane making a constant angle with that axis; and for the same value of n (614) is a conic surface, to which all these planes are tangent. Hence, for the same value of n, (614) is the singular solution, and (612) the complete primitive of (611). But as n is arbitrary, (612) denotes an infinite number of such systems of planes, and (614) an infinite number of such cones; each cone being the assemblage of the characteristics of the system of planes that pass through its vertex. There are, therefore, an infinite number of such systems of planes, each plane of which satisfies the problem; and to each system of planes there is a cone tangent to all the planes of the system, which cone also satisfies the problem.

Equation (612) is, therefore, the most universal solution the problem admits of; and (614), though a singular solution in respect of any one of the system of planes comprised in (612), is, nevertheless, a general solution, inasmuch as it contains an arbitrary constant n.

If the problem proposed were, "*A plane passes through a given point on a fixed line and makes a given angle with the line, deter-*

mine the surface to which the plane is tangent," we would by taking the fixed line for the axis of Z find (612) for the equation of the plane where n is the distance on the axis of Z of the given point from the origin. Eliminating m from this plane by the principle of consecutive surfaces given in the Differential Calculus, we have (614). As the equation (611) is independent of the position of the axis of the cylinder, and is the same whether the cylinder be $x^2 + y^2 = R^2$, or $(x - a)^2 + (y - b)^2 = R^2$, we infer that the systems of surfaces (612), (614), may have any line parallel to the axis of the cylinder, for the line of vertices of the cone, or line of intersections of the systems of planes. Hence, any point in space may be taken as the point through which the system of planes (612) passes. We may express this analytically if we denote by $(a,b,e,)$ the co-ordinates of such a point in space; then for this point (612) becomes,

$$(615) \qquad e = (c^2 - m^2)^{\frac{1}{2}}\, a + mb + n,$$

and subtracting this from (612), we have,

$$(616) \qquad z - e = (c^2 - m^2)^{\frac{1}{2}}\, (x - a) + m\,(y - b).$$

If we eliminate m between (616), and its differential for m variable, we have,

$$(617) \qquad z - e = c\,\sqrt{(x - a)^2 + (y - b)^2},$$

which is a cone with its base parallel to the plane xy.

The same remarks apply to the relation between (616), (617), as were made concerning (612), (614).

It is obvious that (615) furnishes a relation between m and n, in terms, however, of three arbitrary constants a,b,e. If a,b,e, be considered as known, we have simply another condition added to the problem, viz. that the surface which intersects the family of cylinders shall pass through a given point, (a,b,e).

If now we had two of the constants a,b,e, in functions of the third, [which we would have if the point $(a,b,e,)$ were on a curve of double curvature,] we might eliminate this constant and obtain the singular solution under this restriction. But, as there may be an infinite number of curves of double curvature on which the point $(a,b,e,)$ may be situated, there would also be an infinite number of singular solutions of this description.

If any one of the constants a,b,e, be zero, and the other two related as in a plane curve, we can determine the singular solution under this restriction.

This discussion of (612), (614), and (616), (617), exhibits the mode of determining the constants introduced into the integrals of partial differential equations of the first order exceeding the first degree. The nature and conditions of the problem that produced the differential equation are the guide.

The same discussion and remarks apply to Application 47th, Example 2, which proposed to find a surface equivalent to a given plane.

As another example of singular solutions, take the following problems.

APPLICATION 48th.

Through a given point a plane is drawn, cutting off on two of the co-ordinate planes triangles, the sum of whose areas is constant, determine the surface to which this plane is tangent.

APPLICATION 49th.

A plane is drawn tangent to a surface, its traces on the co-ordinate planes form a triangle whose area is given, determine the surface.

APPLICATION 50th.

A plane is drawn tangent to a surface, its traces form with the co-ordinate axes triangles, the sum of whose areas is given, determine the surface.

MISCELLANEOUS PROBLEMS.

PROBLEM A.

A curve is traced on a given cylinder, determine the curve when the cylinder is developed on a plane.

Call the right line which generates the cylinder, *the element of the cylinder.*

Let $(x',y',z',)$ be the co-ordinates of any point in space, then for the element of the cylinder, we have the equations,

(1) $\quad x' - x = a(z' - z),\ $ and $y' - y = b(z' - z).$

Let the given curve traced on the cylinder be represented by

(2) $\quad x = \pi z, \quad y = \Psi z.$

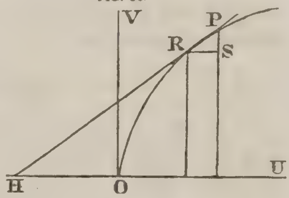
Let the tangent line to this curve be,

(3) $\quad x' - x = p(z' - z), \quad$ and $\quad y' - y = q(z' - z).$

Let P be the angle which the tangent line (3) makes with the element (1), and we have, by Analytical Geometry,

(4) $\quad \text{Cos.P} = \dfrac{1 + ap + bq}{M(1 + p^2 + q^2)^{\frac{1}{2}}}, \quad$ where $M = (1 + a^2 + b^2)^{\frac{1}{2}}.$

FIG. 96.



Suppose now the cylinder be developed, and that OR be the development of the curve. The elements of the cylinder are the ordinates of OR. Let OV, one of these elements, be the axis of V, and call the axis of abscissas the axis of U. It is obvious that the angle P will be the same before and after development. Hence, if HP be the tangent to OR at the point P, and the points P and R, be indefinitely near to each other, we have from the triangle RPS,

(5) $$\text{Cos.P} = \frac{dv}{(du^2 + dv^2)^{\frac{1}{2}}}.$$

Equating (4) and (5), we have,

(6) $$\frac{1 + ap + bq}{M(1 + p^2 + q^2)^{\frac{1}{2}}}, = \frac{dv}{(du^2 + dv^2)^{\frac{1}{2}}}.$$

Also, the length of the element of the curve in space is the same as the length of the element of the curve in the development. Hence, we have,

(7) $$dx^2 + dy^2 + dz^2 = dv^2 + du^2.$$

From the four equations, (2), (6), (7), eliminate x,y,z, the result will be a differential equation between the co-ordinates u and v, which when integrated, may be represented by

(8) $$v = \varphi u.$$

This is the equation of OR, the curve required.

It is obvious that to eliminate x, y, and z, between (2), (6), (7), we must first by means of (2) integrate (7) for one of the co-ordinates x,y,z; and when these co-ordinates are eliminated we will have du and dv, involved in the resulting equation with the sign of integration. This sign of integration being disposed of by differentiation gives a differential equation of the second order whose integral is equation (8).

PROBLEM B.

Given the equation of a plane curve, find the curve when enveloped on a given cylinder.

Let OP, fig. 96, be the plane curve, and (8) its equation.

If the cylinder be placed with its element on OV, and rolled on the plane, the curve OR will be enveloped on the cylinder. Let the equation of the cylinder be

(9) $$F(x,y,z) = o.$$

If now between the equations (6), (7), (8), we eliminate the co-ordinates u and v, the result will be a differential equation which when integrated, we may represent by

(10) $$\varphi(x,y,z) = o.$$

Between (9) and (10), eliminate first y and then x, and we get (2), the equation of the enveloped curve.

If the cylinder be perpendicular to the plane XY, then a and b, in (1) are zero, and (6) is modified accordingly.

If the given curve (8) be a straight line, the angle P is constant, and (6) becomes,

$$(11) \qquad \frac{1 + ap + bq}{M(1 + p^2 + q^2)^{\frac{1}{2}}} = C, \text{ a constant.}$$

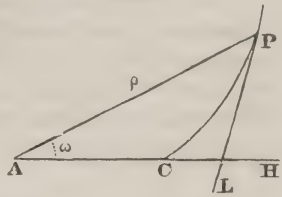
This equation may be at once integrated, and we have (10).

If a and b be zero, (11) becomes the partial differential equation, $p^2 + q^2 = c^2$, whose first integral is (612).

PROBLEM C.

A curve is traced on a given cone, determine the curve when the cone is developed on a plane.

FIG. 97.



Let $(a',b',c',)$ be the vertex of the cone. The equation of the element of the cone is

$$(12) \quad x - a' = a(z - c'), \quad y - b' = b(z - c').$$

Let the curve be,

$$(13) \qquad x = \phi z, \quad y = \Psi z,$$

and let the tangent to (13) be

$$(14) \qquad x' - x = p(z' - z), \quad \text{and} \quad y' - y = q(z' - z).$$

Equation (4) expresses the cosine of the angle made by (12) with (14). As a and b in (12) vary with the position of the element, we must restore to (4) the value of M, and then eliminate a and b from (4), by means of (12), and we have,

$$(15) \qquad \text{Cos.P} = F(x,y,z,p,q).$$

Suppose now the cone developed, and that CP is the curve in the development. Let the vertex of the cone be at A, then the element of the cone becomes the radius vector AP of the developed curve, the pole being the vertex. Then cos. P in the development, is [by Proposition D. 26, Appendix, Differential Calculus,]

$$(16) \qquad \text{Cos.P} = \frac{d\rho}{(d\rho + \rho^2 d\varpi^2)^{\frac{1}{2}}};$$

and since the angle P is the same before and after development, we have by equations (15) and (16),

(17) $$F(x,y,z,p,q) = \frac{d\rho}{(d\rho^2 + \rho^2 d\omega^2)^{\frac{1}{2}}}.$$

Also the length of the element of the given curve being the same before and after development, we have,

(18) $$dx^2 + dy^2 + dz^2 = d\rho^2 + \rho^2 d\omega^2.$$

Eliminate x,y, and z, between the four equations (13), (17) and (18), and we have a differential equation, which when integrated, we may represent by

(19) $$\rho = \varphi\omega.$$

This is the developed curve CP.

If the angle P be constant then (17) becomes,

(20) $$\frac{d\rho}{(d\rho^2 + \rho^2 d\omega^2)^{\frac{1}{2}}} = \mathrm{C},$$

a constant. The integral of this gives us at once, for (19), the equation,

(21) $$\mathrm{Log.}\rho = \frac{\mathrm{C}\omega}{(1 - \mathrm{C}^2)^{\frac{1}{2}}} + \mathrm{C}',$$

the logarithmic curve.

PROBLEM D.

Given the equation of a plane curve, find the curve when enveloped on a given conic surface.

Taking for the conic surface the equation,

(22) $$\varphi(x,y,z) = o,$$

and taking (19) for the given curve, we eliminate ρ and ω between (17), (18), (19), and integrating the resulting equation, we have,

(23) $$\mathrm{F}(x,y,z) = o.$$

Eliminate first y and then x between (22) and (23), and we have (13), the equations of the enveloped curve.

PROBLEM E.

Determine the singular solution of a differential equation without first obtaining the complete primitive.

In Proposition XV, Integral Calculus, we have shown the principle of singular solutions of differential equations, on the supposition that we have the complete primitive of the equation.

It may be convenient to deduce a method for obtaining such solutions when they exist without seeking first the integral of the equation

Suppose we have the primitive equation,

$$(24) \qquad F(x,y,c) = o = u,$$

where c is the arbitrary constant, suppose that c enters into (24) in such a manner that it remains in the immediate differential of (24), then if

$$(25) \qquad Mdx + Ndy = o,$$

be the immediate differential of (24), we may get a differential equation by eliminating c between (24) and (25). Suppose the value of c from (25), be represented by

$$(26) \qquad c = \varphi(x,y,p).$$

Put this value of c into (24), and we have,

$$(27) \qquad f(x,y,\varphi) = o = u',$$

where, for brevity, φ is put for $\varphi(x,y,p)$.

It is evident that equation (24) will be produced by integrating either (27) or (25).

If (24) be solved for y, we may represent the result by

$$(28) \qquad y = f(x,c);$$

and when this admits of a singular solution, we have, as in Proposition XV,

$$(29) \qquad \frac{dy}{dc} = o,$$

from which we deduce $c = f(x)$, and eliminating c between (29) and (28), we get the singular solution sought.

Let us now without obtaining its primitive (24), examine to what conditions the differential equation (27) is subject when it admits of a singular solution. For this purpose differentiate (27), and for convenience represent its differential [since ϕ contains x,y and p] by

$$(30) \qquad du' = Pdx + Qdy + Rdp = o,$$

where P, Q, R, are functions of x,y, and p. Now regarding (28)

as the integral of (27), we have y, a function of x and c, consequently p is also a function of x and c, and when (27) admits of a singular solution, c may be taken variable in (28), because in (29) we have $c = fx$. Hence, in order that (30) may be the complete differential of (27), we must put into (30) for dy and dp, their values in respect both of x and c. These values are,

$$(31) \qquad dy = \frac{dy}{dx}dx + \frac{dydc}{dc}, \quad \text{and} \quad dp = \frac{dpdy}{dy} + \frac{dpdc}{dc}$$

which being put into (30), that equation becomes,

$$(32) \quad du' = \left(P + pQ + R\frac{dp}{dx}\right)dx + \left(Q\frac{dy}{dc} + R\frac{dp}{dc}\right)dc = o.$$

This is the differential of (27) for x,y and c variable.

The term in (32) which is multiplied by dx, is evidently the same as equation (30) which being identically zero, independently of c, equation (32) reduces to

$$(33) \qquad \left(Q\frac{dy}{dc} + R\frac{dp}{dc}\right)dc = o.$$

This is satisfied by making $dc = o$, or,

$$(34) \qquad Q\frac{dy}{dc} + R\frac{dp}{dc} = o.$$

The condition $dc = o$ gives $c =$ a constant, which does not make known any thing concerning (27). The condition (34) being examined, we observe that in case (27) admits of a singular solution, the condition (29) exists, which reduces (34) to

$$(35) \qquad R\frac{dp}{dc} = o,$$

which is satisfied by putting,

$$(36) \qquad \frac{dp}{dc} = o, \quad \text{or,} \quad R = o.$$

The condition that gave us (33) was,

$$(37) \qquad du' = P + pQ + R\frac{dp}{dx} = o,$$

which by virtue of the condition $R = o$, becomes

$$(38) \qquad P + pQ = o.$$

Now, though c does not enter expressly into (27), or its differential (30), yet it is virtually contained in these equations, because it is the value of φ which enters these equations, consequently, it may be regarded as existing implicitly in (30). The second condition of (36); viz. $R = o$, must, therefore, exist in the case of a singular solution. This condition, and (38) may, therefore, be deduced immediately from (27). For if we solve (37) for the coefficient of R, we have,

$$(39) \qquad \frac{dp}{dx} = -\frac{P + pQ}{R},$$

which by conditions (36) and (38), becomes

$$(40) \qquad \frac{dp}{dx} = \frac{o}{o}.$$

Hence in case (27) admits of a singular solution, (27) must be such that if we differentiate it for x,y and p variable, and deduce from this differential the value of $\frac{dp}{dx}$, we will have a fraction whose numerator and denominator being each put equal to zero, will be the conditions of the singular solution.

These conditions, viz. (38) and the last of (36), furnish two equations which with each other, or with the given differential equation (27), will enable us to eliminate p. The result, which may be represented by

$$(41) \qquad \phi(x,y) = o,$$

will be the singular solution of (27), provided it satisfies that equation. If the differential coefficient from (41) put into (27) does not render (27), when combined with (41), an identical equation, we conclude that (27) does not admit of a singular solution.

In this procedure we have supposed the differential equation (27) freed from radicals. If that equation appears in the surd form, it should be first rationalised by squaring, &c., before it is differentiated.

We will give a few examples of the determination of singular solutions by this theory.

Ex. 1. Determine the singular solution of the equation

(a) $$(x + y)\, p - p^2x - (a + y) = o.$$

Differentiating this for x, y and p variable, we get

(b) $$\frac{dp}{dx} = \frac{p^2 + p - p - p^2}{x + y - 2px}.$$

The numerator of this is identically zero, and the denominator being put equal to zero, we have,

(c) $$x + y - 2px = o.$$

Eliminate p between (a) and (c), and we have,

(d) $$(x - y)^2 = 4ax.$$

This satisfies the given differential equation (a). Hence, (d) is the singular solution of (a).

Ex. 2. Find a curve whose normal varies as the part of the axis intercepted between it and the origin.

By the enunciation we have, putting n for the ratio of the normal to the intercept,

(e) $$y^2(1 + p^2) = n^2(x + py)^2.$$

To determine whether this has a singular solution, we differentiate it as before, and putting numerator and denominator of $\frac{dp}{dx}$, equal to zero, we find,

(f) $$y = px,$$

and eliminating p between (f) and (e), we find,

(g) $$y = \pm \frac{nx}{(1 - n^2)^{\frac{1}{2}}}.$$

This satisfies (e), and is, therefore, the singular solution.

Ex. 3. Find the curve whose normal varies as the square root of the part of the axis intercepted between it and the origin. By the proposition, we have,

(h) $$y^2(1 + p^2) = n^2(x + py).$$

Proceeding as before, we find for the singular solution, a parabola.

Ex. 4. Given the equation.

$$y = px + (1 + p^2)^{\frac{1}{2}}x.$$

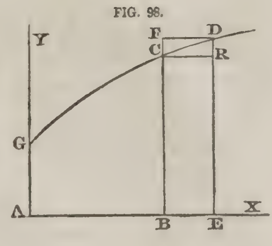
Prove that it has no singular solution.

PROBLEM F.

Determine the area of a plane curve.

Proposition I. of the text exhibits the mode of determining the differential of a plane area by the method of indefinitely small quantities. To deduce the differential of the same area by the method of limits proceed as follows:

FIG. 98.



Let GD be the curve and let the area sought be the part AGCB between the curve and the axis AX.

Put the area AGCB = A, CB = y, AB = x, BE = h. Complete the rectangles BR and BD.

By Taylor's Theorem we have for the ordinate DE the value

$$\text{DE} = y + \frac{dy}{dx} h + \frac{d^2y}{2dx^2} h^2 + \&\text{c.}$$

The space DCBE is the increment of the area AGCB corresponding to the increment h of the abscissa. This space DCBE is contained between the rectangles BR and BD. The areas of these rectangles are

$$\text{BR} = hy, \text{BD} = h\left(y + \frac{dy}{dx} h + \frac{d^2y}{2dx^2} h^2 + \&\text{c.}\right)$$

DCBE is therefore contained between the values

$$(m) \qquad hy \quad \text{and}$$

$$(n) \qquad h\,(y + \frac{dy}{dx} h + \frac{d^2y}{2dx^2} h^2 + \&\text{c.})$$

But in the limit, the development

$$(o) \qquad y + \frac{dy}{dx} h + \frac{d^2y}{2dx^2} h^2 + \&\text{c.},$$

becomes simply y; and (m) and (n) are equal, and the space DCBE becomes the differential of the area which is denoted by d.A. Hence the differential of the area being contained between the values (m) and (n) and these being in the limit equal, the differential of the area is equal to the value (m) and we have

$$(p) \qquad d.\text{A} = yh = ydx, \qquad \text{or integrating}$$

$$(q) \qquad \text{A} = \int ydx.$$

Equations (p) and (q) are the same as found in Proposition I. of the text and are employed in the same manner as the equations of that Proposition.

By similar reasoning we might proceed to determine by the method of limits, the properties of lines, surfaces, and solids which are presented in Proposition III. to XI. of the Integral Calculus. The results of the investigation would in all cases coincide with those presented in the text. Want of space however precludes us from giving more than the foregoing example of the method.

THE END.

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From Peter S. Duponceau, President of the American Philosophical Society.

PHILADELPHIA, *Jan. 18th*, 1844.

GENTLEMEN,—I beg you will accept my thanks for the honour done me, by presenting me with a copy of your improved edition of *Fleming and Tibbins' French and English and English and French Dictionary;* such a work was really wanting in our literature. All the English and French and French

and English Dictionaries that have hitherto appeared, have been compiled, as far as I know, by natives of France, among whom Boyer and Chambaud are the most distinguished. They do not appear to have had the aid of natives of the British isles, whose language of course was not so familiar to them as their own. Such dictionaries, to be perfect, ought to have been the joint labour of natives of the two countries. That defect is now corrected by the work of two English authors, teachers and resident in Paris, founded on that of the best of their French predecessors. So that it unites the knowledge of the best lexicographers of the two nations. This is an immense advantage, which cannot be overlooked.

In addition to this, it has been justly observed, that great changes have taken place in the two languages since the beginning of the present century. It is a curious fact that the two idioms have in a great measure migrated, and are still migrating into each other. It is the result of the great intercourse which, during the last twenty-five years, has taken place between the two nations. These changes are not noticed in any other existing bilingual Dictionary. This therefore is an advantage possessed only by the Dictionary you have published. The additions by Messrs. Dobson and Picot, are very valuable, and assure the superiority of your edition. I hope that it will meet with the success that it deserves.

am, respectfully,
your most obedient servant,
PETER S. DUPONCEAU.

From Robley Dunglison, M. D., Professor of Materia Medica and Practice of Medicine in the Jefferson Medical College, and Secretary of the American Philosophical Society.

The undersigned having been requested to express his opinion of the merits of the *New and Complete French and English Dictionary*, of Messrs. Fleming and Tibbins, with the additions made to it by Messrs. Dobson and Picot, of this city, has no hesitation in expressing his decided conviction, that it is the most comprehensive and satisfactory Dictionary of the French and English languages combined, that could be put into the hands of the young student of those languages; whilst it is, at the same time, the best accompaniment to him who is farther advanced.

ROBLEY DUNGLISON, M. D.

From F. A. Brégy, Professor of the French and Spanish Languages in the Central High School.

CENTRAL HIGH SCHOOL,
PHILADELPHIA, *Jan. 20th*, 1847.

GENTLEMEN,—Having been requested by you, to express my opinion as to the merits of Fleming and Tibbins' French and English Dictionary, I cannot but concur in the decided approbation which it received from many eminent French and classical scholars, at the time of its first importation into this country. I have given it a close and careful examination, and have used it for two years in my classes in this institution. The result of this trial and examination has been a settled conviction, that the work of Fleming and Tibbins is by far superior to its predecessors in the same line. Such a work was greatly needed by the friends of the French language and literature in America, on account of the great number of new words used by modern writers, which present to the inexperienced reader difficulties not easily solved with the aid of Meadows, Boyer, and others. The merit of these works *has been* very great indeed, but they have grown *too old*. Let it suffice to state that the single letter *A* contains in Fleming and Tibbins *abridged*, several hundreds of words more than the corresponding

letter in some of the others. I do not hesitate, therefore, to recommend the work above alluded to, as indispensably necessary to American students desirous of understanding not only the French of the 17th and 18th century, but also that of the 19th, that of Chateaubriand, Lamennais, Victor Hugo, Thiers, and of the host of scientific as well as literary French writers of the present time.

I am very respectfully yours,

F. A. BREGY.

BOSTON, *January* 22, 1844.

Ever since the first importation of Fleming and Tibbins' French and English Dictionary, I have constantly had it on my table, and have found it better than all other French dictionaries. I am therefore rejoiced to see an American abridged edition of so excellent a work. I find that all which is most essential in the French edition is retained, and many decided and valuable improvements are made. The mode in which the pronunciation is indicated is admirably plain and thorough; a vast number of words not to be found in other dictionaries is introduced; an excellent arrangement of the verbs is given; and it is printed in a large and easily legible type. Altogether, it is decidedly the best French dictionary I have seen.

Respectfully yours,

GEORGE B. EMERSON.

BOSTON HIGH SCHOOL,
February 1, 1843.

GENTLEMEN,—I have devoted some time and attention to the examination of Fleming and Tibbins' French and English Dictionary, lately published by your firm; and, although the merits of such a work can be thoroughly tested only by long use and a careful collation of it with kindred works, yet I must say that this dictionary bears evident marks of its superiority to any other that has been introduced into this country.

By comparison, I find its vocabulary very copious and the idiomatic phrases quite numerous. The technical terms are a very important addition, and the conjugation of verbs will prove of great use to the learner. The mechanical execution of the work, which is highly important in a dictionary, is a recommendation which immediately impresses itself on the eye.

A complete and accurate dictionary is of the utmost importance in the acquisition of a foreign language, and I feel justified in recommending this as one of great excellence

Very respectfully, yours,

THOMAS SHERWIN.

From Isaac Leeser, Minister of the Hebrew Portuguese Congregation, Philadelphia.

GENTLEMEN—It is with much pleasure that I have perceived the publication of Fleming and Tibbins' Dictionary of the French language. During its progress through the press I have had occasion to look over several parts thereof, and I became convinced that it would prove an invaluable aid to those who wish to acquire a knowledge of the most fashionable language of Europe. To its original and intrinsic merit is to be superadded the additions of the American editor, who has enriched it with more than *five thousand* words (Medical, Botanical, &c., &c.,) not in the French copy; also an excellent table of verbs, furnished by Mr. Picot. I cannot doubt that it will soon become an especial favourite with a discerning public; especially, as the moderate price you have fixed on it, a little more than one-fourth of the cost of the Paris edition, will bring this valuable Lexicon within the reach of the general student. Respectfully yours,

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RECOMMENDATIONS.

From Professor C. D. Cleveland, Philadelphia.

The publisher requests my opinion of Parley's Common School History. It is seldom that I give an opinion upon school books, there are so few that I can recommend with a clear conscience; and publishers do not wish, of course, to send forth a condemning sentence to the world. But in this case I can truly say that, having used the book in my school since it was published, I consider it a most interesting and luminous compend of general history for the younger classes of scholars; and that, were I deprived of it, I know not where I could find a work that I could use with so much pleasure to myself, and profit to those for whom it is designed.

Respectfully yours,
C. D. CLEVELAND.

I fully concur in the above opinion of Professor Cleveland.
JOHN FROST,
Professor of Belles Lettres in the High School of Philadelphia.

From Rev. C. H. Alden, late Principal of the High School for Young Ladies, Philadelphia; now Chaplain in the U. S. Navy.

I am greatly pleased with Parley's Universal History, and shall introduce it into my school. Judicious in its arrangements, attractive in style, and striking in selection, it commends itself to teachers and parents as particularly appropriate to the juvenile mind. When known to school committees, and others concerned with our common schools, it cannot fail of being introduced into general use. The Geographical and Chronological accompaniments greatly enhance the value of the work; nor should the numerous and appropriate cuts and maps be overlooked, furnishing, as they do, a very valuable addition to both its beauty and utility.

Philadelphia, September 19, 1839.

Having examined Parley's Common School History, I do not hesitate to say that, in my opinion, it is decidedly the best elementary general history I have seen, and I recommend its use to other teachers.
M. L. HURLBUT.

The above is concurred in by the undersigned, as follows:

I intend to introduce it into the academical department of the University of Pennsylvania, under my care, as soon as possible.
SAMUEL W. CRAWFORD.

I have already introduced Parley's Common School History as a class-book.
SAMUEL JONES,
Principal of Classical and Mathematical Institute.

After a careful examination of the above-mentioned work, I am convinced that it is the best treatise for beginners in history, whether juvenile or adult, that I have ever seen.
J. J. HITCHCOCK.

I fully concur in the above.
R. P. HUNT.

I consider the work as a valuable acquisition in our schools for elementary classes in History.
M. SEMPLE, M. D.,
Principal of Young Ladies' Institute.

Parley's Common School History will, in my opinion, prove a valuable work in every school in which history is a subject of instruction. The modesty of its title is far from giving an adequate idea of its excellence.
OLIVER A. SHAW.

The following resolution passed the Board of Controllers of the Public Schools of Philadelphia, September 9, 1841:

"*Resolved*, That the teachers of the different schools may, with the approval of the various Visiting Committees, establish, where the same has not already been done, History classes; and that they use for the pupils in said classes, 'Parley's Common School History,' in addition to any other history heretofore authorized by the Board;—said book to be supplied at the cost of the respective pupils."

THOMAS B. FLORENCE,
From the Minutes. *Secretary*.

Philadelphia, April 30, 1842.

MR. E. H. BUTLER.

Sir,—I have with much pleasure given to Parley's Common School History a very thorough examination. Its object being to present to the youthful mind a brief and interesting view of the world's history, as might be expected from the distinguished reputation of its author, this end has been admirably attained. The most remarkable events which have occurred since the creation of the world, have been carefully selected; and have been described in so perspicuous a manner, that even beginners in the study of history can readily comprehend them. So felicitous are the descriptions, and such is the interest with which the author has succeeded in investing the subject, that the labour of study is converted into pleasure: and the impressions made, are not easily effaced.

Yours truly,
SAMUEL RANDALL.
Young Ladies' Seminary, 229 *and* 231 *Arch street*

We fully concur in the above commendation of Parley's Common School History, having used it with much satisfaction in our schools.

P. A. CREGAR,
D. R. ASHTON,
W. BRADFORD,
WILLIAM NOTSON,
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JOHN MOONEY,
E. ROBERTS,
D. M. PADDACK,
C. PELTON,
JOSEPH P. ENGLES,
THOMAS M'ADAM,
JOHN EVANS,
L. W. BURNET,
Teacher, Fourth and Vine streets.

Philadelphia, May 5, 1842.

A careful examination of "Parley's Common School History," gave me a very favourable opinion of its excellence; and the *trial* which I have since made of it as a class-book in my school, has convinced me that I did not overrate its merits.

JOHN ALLEN,
Principal of a Female Seminary, No. 274 *North Seventh street.*

I feel much interested in the volumes which were left for my examination. Parley's Common School History is excellent in its way; the engravings serve to explain the dress, armour, &c. of former years to youth, who receive a more decided and satisfactory explanation by means of a cut, than by pages of description; the maps are equally appropriate.

I shall have no difficulty in forming a class of physiology, my pupils take hold of it with eagerness; and treated so delicately, as it is in this book, I can see no objection to its introduction, while some knowledge of this subject, amidst the onward progress of education, will soon be regarded as indispensable.

W. CURRAN.

Baltimore.

Having examined Parley's Common School History, I do not hesitate to say that, in my opinion, it is decidedly the *best* elementary General History I have seen, and I recommend its use to other teachers.

A. B. CLEVELAND,
Female Classical School.

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The arrangement of the several parts of the work is strictly systematic; each is discussed in its natural order, and with as much brevity as consists with perspicuity. The work is designed for the use of colleges and schools, as well as for the instruction of private individuals who desire to improve themselves in the art of reading and speaking.

The value of vocal gymnastics cannot be duly appreciated by those who have not experienced or witnessed their beneficial results. But the time is not far distant when these exercises will be considered, by all intelligent persons, an essential part of primary instruction.

From Benjamin Hallowell, Principal of the Friends' High School, Philadelphia.

I have examined with some care the "First Lines of Natural Philosophy," by Reynell Coates, M.D., published by E. H. Butler & Co., and I believe it to be a work unusually well calculated to give to the student a clear understanding of the different subjects upon which it treads. The clear and familiar style of the author, and his mode of preparing the learner to comprehend the meaning of the different scientific terms, previous to their being used in the work, are advantages, in my opinion of no small merit.

Signed,

BENJAMIN HALLOWELL.

Philadelphia, 11th mo. 20th, 1845.

From the Presbyterian.

We have seldom been more gratified by the execution of an elementary book, than by the one before us. The intelligent reader will at once perceive that it is not a mere compilation, in which the materials are well selected and arranged, but the product of a philosophical mind—the result of much thought and skilful analysis. The author displays a clear comprehension of his subject, and he has succeeded remarkably in communicating an intelligible view of it to his reader. In our opinion he has furnished one of the best text books we have met with, for the aid of those who wish to obtain a good general view of this interesting science.

From the Alexandria (Va.) Gazette.

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From the Philadelphia Ledger.

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M. L. HURLBUT.

I concur in the above opinion of Mr Hurlbut.

JOHN FROST.

Philadelphia, April 30, 1842.

MR. E. H. BUTLER.

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Yours, truly,

SAMUEL RANDALL,
Young Ladies' Seminary, 229 *and* 231 *Arch Street.*

May 5, 1842.

I concur in the above expression of Samuel Randall, Esq., and trust that the community will encourage the use of Dr. Coates's work.

JOHN D. BRYANT,
Principal of English and Classical Academy.

I most heartily concur with the above recommendation.

S. W. CRAWFORD.

So far as we have had time to examine Dr. Coates's Physiology, we agree with above recommendations.

W. BRADFORD.

L. W. BURNET,
Teacher at corner 4th and Vine Streets.

We, the undersigned, fully concur with the opinion expressed in the above recommendation.

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D. R. ASHTON,
THOMAS BALDWIN,
WILLIAM NOTSON,
SPENCER ROBERTS,
JOHN MOONEY,
E. ROBERTS,
D. M. PADDACK,
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JAMES GOODFELLOW,
J. WOOD,
JOSEPH P. ENGLES,
THOMAS M'ADAM,
JOHN EVANS,
THOMAS T. AZPELL,
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Philadelphia, May 7, 1842.

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Central High School,
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